Recent Developments on Multiple Eisenstein Series in Positive Characteristic

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Outline

Let

$$\mathcal{I} = \bigcup_{m=1}^{\infty} \mathbb{N}^m \cup \{\varnothing\}$$

be the set of indices. For $\mathfrak{a} = (a_1, \ldots, a_m) \in \mathcal{I}$, we define

$$\mathrm{wt}(\mathfrak{a}) = \sum_{i=1}^{m} a_i \quad \text{and} \quad \mathrm{dep}(\mathfrak{a}) = m.$$

We recall the **multiple zeta values (MZVs)**.

Definition 1.1

Let $\mathfrak{a} = (a_1, \ldots, a_m) \in \mathcal{I}$ be an admissible index, i.e., $a_1 \geq 2$. Define

$$\zeta(\mathfrak{a}) = \sum_{n_1 > \cdots > n_m \geq 1} \frac{1}{n_1^{a_1} \cdots n_m^{a_m}} \in \mathbb{R}.$$

It is known that for any admissible $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$,

$$\zeta(\mathfrak{a})\zeta(\mathfrak{b}) \in \mathrm{span}_{\mathbb{Q}}\{\zeta(\mathfrak{c}) \mid \mathrm{wt}(\mathfrak{c}) = \mathrm{wt}(\mathfrak{a}) + \mathrm{wt}(\mathfrak{b}), \mathfrak{c} \in \mathcal{I} \text{ is admissible}\}.$$

In fact, this product admits two interpretations:

- the **stuffle relations**, coming from the series expression.
- the **shuffle relations**, coming from the integral expression.

Example 1.2 (Stuffle Relations)

For $a, b > 1$, we have

$$\begin{aligned}\zeta(a)\zeta(b) &= \sum_{n,m\geq 1} \frac{1}{n^a m^b} = \left(\sum_{n>m\geq 1} + \sum_{m>n\geq 1} + \sum_{m=n\geq 1} \right) \frac{1}{n^a m^b} \\ &= \zeta(a,b) + \zeta(b,a) + \zeta(a+b).\end{aligned}$$

Example 1.3 (Shuffle Relations)

We have

$$\zeta(2)\zeta(2) = \left(\int_{1>x_1\geq x_2>0} \frac{dx_1}{x_1}\frac{dx_2}{1-x_2}\right)\left(\int_{1>y_1\geq y_2>0} \frac{dy_1}{y_1}\frac{dy_2}{1-y_2}\right)$$
$$= \int_{\substack{1>x_1\geq x_2>0\\ 1>y_1\geq y_2>0}} \frac{dx_1}{x_1}\frac{dx_2}{1-x_2}\frac{dy_1}{y_1}\frac{dy_2}{1-y_2}.$$

Decomposing the domain of integration, this becomes

$$4\zeta(3,1)+2\zeta(2,2).$$

It follows immediately that

$$\mathcal{Z} = \operatorname{span}_{\mathbb{Q}}\{\zeta(\mathfrak{a}) \mid \mathfrak{a}\in\mathcal{I} \text{ is admissible}\}$$

forms a $\mathbb{Q}$-algebra.

Let $\tau \in \mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. We define a partial order $\succ$ on the lattice $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ as follows:

- $\lambda = m\tau + n \succ 0$ if $m > 0$, or if $m = 0$ and $n \in \mathbb{N}$.
- $\lambda_1 \succ \lambda_2$ if $\lambda_1 - \lambda_2 \succ 0$.

We recall the **multiple Eisenstein series (MES)**.

Definition 1.4 (Gangl–Kaneko–Zagier, 2006; Bachmann, 2012)

For $\mathfrak{a} = (a_1, \ldots, a_m) \in \mathcal{I}$ with $a_1, \ldots, a_m \geq 2$, we define

$$G(\mathfrak{a};\tau) = \sum_{\substack{\lambda_1,\ldots,\lambda_m \in \Lambda_\tau \\ \lambda_1 \succ \cdots \succ \lambda_m \succ 0}} \frac{1}{\lambda_1^{a_1} \cdots \lambda_m^{a_m}} \in \mathcal{O}(\mathfrak{H}).$$

In the case $a_1 = 2$, we use Eisenstein summation.

Similarly to the classical case where even zeta values appear in the constant term of Eisenstein series, we have the following result.

Theorem 1.5 (Gangl–Kaneko–Zagier, 2006; Bachmann, 2012)

For $\mathfrak{a} = (a_1, \ldots, a_m) \in \mathcal{I}$ with $a_1, \ldots, a_m \geq 2$, we have

$$G(\mathfrak{a}; \tau) = \zeta(\mathfrak{a}) + \sum_{n=1}^{\infty} c_n q^n$$

where $q = e^{2\pi i \tau}$, and each coefficient c_n can be written explicitly in terms of MZVs and generalized divisor functions.

A natural question arises:

Question

Do MES satisfy the same stuffle and shuffle relations?

Example 1.6

In the shuffle relation

$$\zeta(3)\zeta(3) = 12\zeta(5,1) + 6\zeta(4,2) + 2\zeta(3,3),$$

the MES for $\mathfrak{a} = (5,1)$ is not defined.

This shows that a naive extension of the stuffle and shuffle relations to MES may fail.This obstacle can be overcome by introducing suitable regularizations:

- (Bachmann–Tasaka 2018) Introduced the **shuffle-regularized MES** $G^{\sqcup\!\sqcup}(\mathfrak{a})$ and showed that they satisfy the shuffle relations.
- (Bachmann 2019) Introduced the **stuffle-regularized MES** $G^{*}(\mathfrak{a})$ and showed that they satisfy the stuffle relations.

Let p be a prime and let q be a power of p. We consider

$$\begin{array}{ccccccc}
A := \mathbb{F}_q[\theta] & \subseteq & K := \mathbb{F}_q(\theta) & \subseteq & K_\infty := \mathbb{F}_q((1/\theta)) & \subseteq & \mathbb{C}_\infty := \widehat{\overline{K_\infty}} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\mathbb{Z} & \subseteq & \mathbb{Q} & \subseteq & \mathbb{R} & \subseteq & \mathbb{C}
\end{array}$$

We let $A_+ \subseteq A$ be the set of monic polynomials, which plays the role of the positive integers $\mathbb{N}$. We recall **Thakur's multiple zeta values.**

Definition 2.1 (Thakur, 2004)

Let $\mathfrak{a} = (a_1, \ldots, a_m) \in \mathcal{I}$. Define

$$\zeta_A(\mathfrak{a}) = \sum_{\substack{f_1,\ldots,f_m \in A_+ \\ \deg f_1 > \cdots > \deg f_m \geq 0}} \frac{1}{f_1^{a_1} \cdots f_m^{a_m}} \in K_\infty.$$

Similar to the classical case, we have the *q*-**shuffle relations**.

- (Thakur, 2009) For $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$,

$$\zeta_A(\mathfrak{a})\zeta_A(\mathfrak{b}) \in \operatorname{span}_{\mathbb{F}_p}\{\zeta_A(\mathfrak{c}) \mid \operatorname{wt}(\mathfrak{c}) = \operatorname{wt}(\mathfrak{a}) + \operatorname{wt}(\mathfrak{b}), \mathfrak{c} \in \mathcal{I}\}$$

  and hence $\mathcal{Z} = \operatorname{span}_{\mathbb{F}_p}\{\zeta_A(\mathfrak{a}) \mid \mathfrak{a} \in \mathcal{I}\}$ forms an $\mathbb{F}_p$-algebra.

- (Chen, 2015) Gave an explicit product formula: For any $a, b \in \mathbb{N}$,

$$\zeta_A(a)\zeta_A(b) = \zeta_A(a,b) + \zeta_A(b,a) + \zeta_A(a+b) + \sum_{\substack{i+j=a+b \\ (q-1)\mid j}} \Delta_{a,b}^{i,j}\zeta_A(i,j)$$

  where $\Delta_{a,b}^{i,j} = (-1)^{a-1}\binom{j-1}{a-1} + (-1)^{b-1}\binom{j-1}{b-1} \in \mathbb{F}_p$.

- (Observed by Yamamoto; formulated by Shi, 2018) Define the q-**shuffle algebra** $(\mathcal{R}, *)$, where

$$\mathcal{R} = \mathrm{span}_{\mathbb{F}_p}\{x_{\varnothing} = 1, x_{\mathfrak{a}} = x_{a_1} \cdots x_{a_m} \mid \mathfrak{a} = (a_1, \ldots, a_m) \in \mathcal{I}\}$$

and the q-**shuffle product** $*$ is defined recursively by:

- For all $X \in \mathcal{R}$,

$$1 * X = X * 1 = X$$

.

- For all $a, b \in \mathbb{N}$ and $X, Y \in \mathcal{R}$,

$$x_a X * x_b Y = x_a(X * x_b Y) + x_b(x_a X * Y) + x_{a+b}(X * Y) + \sum_{\substack{i+j=a+b \\ (q-1)|j}} \Delta_{a,b}^{i,j} x_i((X * Y) * x_j).$$

We notice that the q-shuffle product $*$ is commutative.

Shi showed that the *q*-shuffle algebra $(\mathcal{R}, *)$ encodes the *q*-shuffle relations.

Theorem 2.2 (Shi, 2018)

The unique $\mathbb{F}_p$-linear map

$$\widehat{\zeta_A} : \mathcal{R} \longrightarrow K_\infty, \qquad x_{\mathfrak{a}} \longmapsto \zeta_A(\mathfrak{a}) \quad (\mathfrak{a} \in \mathcal{I}),$$

is an $\mathbb{F}_p$-algebra homomorphism. That is,

$$\zeta_A(\mathfrak{a}) \cdot \zeta_A(\mathfrak{b}) = \widehat{\zeta_A}(x_{\mathfrak{a}}) \cdot \widehat{\zeta_A}(x_{\mathfrak{b}}) = \widehat{\zeta_A}(x_{\mathfrak{a}} * x_{\mathfrak{b}})$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$.

This allows us to compute the q-shuffle relations recursively using the inductive formula for the q-shuffle product $*$.

To define MES in this setting, we introduce some necessary notions:

Definition 2.3

Let $r \geq 1$. The **Drinfeld symmetric space of rank** r Ω^r consists of elements in $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$ with K_∞-linearly independent entries.

- We viewed Ω^r as a subset of $\mathbb{C}_\infty^r$ by normalizing the last coordinate to 1.
- Ω^r admits a rigid analytic structure.
- We denote by $\mathcal{O}(\Omega^r)$ the algebra of rigid analytic functions on Ω^r.
- There is a natural $\mathrm{GL}_r(A)$-action on Ω^r.

$\mathbb{H}$	$\longleftrightarrow$	Ω^r
$\tau = [\tau : 1] \in \mathbb{H}$	$\longleftrightarrow$	$\mathbf{z} = [z_1 : \cdots : z_{r-1} : 1] \in \Omega^r$
$\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$	$\longleftrightarrow$	$\Lambda_{\mathbf{z}} = Az_1 + \cdots + Az_{r-1} + A$

In what follows, we let $\mathbf{z} = (z_1; \mathbf{z}') \in \Omega^r$ for $r \geq 2$. We define

$$\exp_{\Lambda_{\mathbf{z}}}(X) = X \prod_{\lambda \in \Lambda_{\mathbf{z}} \setminus \{0\}} \left(1 - \frac{X}{\lambda}\right) \quad \text{and} \quad t_{\Lambda_{\mathbf{z}}}(X) = \sum_{\lambda \in \Lambda_{\mathbf{z}}} \frac{1}{X + \lambda}.$$

Then we have the following theorem:

Proposition 2.4 (Basson-Breuer-Pink, 2024; Gekeler, 2025)

Let $r \geq 2$ and $F \in \mathcal{O}(\Omega^r)$ with $(\gamma \cdot F)(\mathbf{z}) = F(\mathbf{z})$ for all $\gamma = \left(\begin{array}{c|c} 1 & * \\ \hline 0 & I_{r-1} \end{array}\right) \in \mathrm{GL}_r(A)$. Then we have the unique expansion

$$F(\mathbf{z}) = \sum_{n \in \mathbb{Z}} F_n(\mathbf{z}') t_{\Lambda_{\mathbf{z}'}}(z_1)^n$$

with $F_n(\mathbf{z}') \in \mathcal{O}(\Omega^{r-1})$ in a neighborhood of infinity.

For $\mathbf{f} = (f_1, \ldots, f_r) \in A^r$ and $\mathbf{z} = (z_1, \ldots, z_r) \in \Omega^r$, we put

$$\langle \mathbf{f}, \mathbf{z} \rangle = \sum_{i=1}^{r} f_i z_i \in \Lambda_{\mathbf{z}} = Az_1 + \cdots + Az_{r-1} + A.$$

Then we define a partial order $\succ$ on the lattice $\Lambda_{\mathbf{z}}$ as follows:
Let $\langle \mathbf{f}, \mathbf{z} \rangle$ and $\langle \mathbf{g}, \mathbf{z} \rangle \in \Lambda_{\mathbf{z}}$.

- $\langle \mathbf{f}, \mathbf{z} \rangle \succ 0$ if $f_1 = \cdots = f_{i-1} = 0$ and $f_i \in A_+$ for some $1 \leq i \leq r$.
- For $\langle \mathbf{f}, \mathbf{z} \rangle, \langle \mathbf{g}, \mathbf{z} \rangle \succ 0$, we write $\langle \mathbf{f}, \mathbf{z} \rangle \succ \langle \mathbf{g}, \mathbf{z} \rangle$ if one of the following holds:
    - There exists $1 \leq i \leq r$ such that $f_1 = \cdots = f_{i-1} = g_1 = \cdots = g_{i-1} = 0$ and $f_i, g_i \in A_+$ with $\deg f_i > \deg g_i$.
    - There exists $1 \leq i \leq r$ such that $f_1 = \cdots = f_{i-1} = g_1 = \cdots = g_{i-1} = 0$ and $f_i \in A_+, g_i = 0$.

The the **multiple Eisenstein series of rank** r are defined as follows:

Definition 2.5 (Chang–Chen–Huang–T., 2025)

Let $r \geq 1$ and $\mathfrak{a} = (a_1, \ldots, a_m) \in \mathcal{I}$. We define

$$E_r(\mathfrak{a}; \mathbf{z}) = \sum_{\substack{\mathbf{f}_1, \ldots, \mathbf{f}_m \in A^r \\ \langle \mathbf{f}_1, \mathbf{z} \rangle \succ \cdots \succ \langle \mathbf{f}_m, \mathbf{z} \rangle \succ 0}} \frac{1}{\langle \mathbf{f}_1, \mathbf{z} \rangle^{a_1} \cdots \langle \mathbf{f}_m, \mathbf{z} \rangle^{a_m}} \in \mathcal{O}(\Omega^r).$$

Remark 2.6

Double Eisenstein series of rank 2 were first introduced by Chen in 2017. Furthermore, "twisted" vectorial multiple Eisenstein series of rank 2 were considered by Pellarin in his 2025 Memoirs paper.

Notice that $\Omega^1 = \{1\}$ and hence $E_1(\mathfrak{a}; \mathbf{z}) = \zeta_A(\mathfrak{a})$.

Definition 2.7 (Goss, 1980; Gekeler 1988)

Let $G_n^{\Lambda_{\mathbf{z}}}(X) \in \mathcal{O}(\Omega^r)[X]$ be the **Goss polynomials** defined by

$$G_1^{\Lambda_{\mathbf{z}'}}(X) = X \quad \text{and} \quad G_n^{\Lambda_{\mathbf{z}}}(X) = X \sum_{i=0}^{\lceil \log_q n \rceil - 1} \alpha_i(\mathbf{z}) G_{n-q^i}^{\Lambda_{\mathbf{z}}}(X),$$

where $\exp_{\Lambda_{\mathbf{z}}}(X) = \sum_{i=0}^{\infty} \alpha_i(\mathbf{z}) X^i$. Then we have the following:

- $G_n^{\Lambda_{\mathbf{z}}}(t_{\Lambda_{\mathbf{z}}}(X)) = \sum_{\lambda \in \Lambda_{\mathbf{z}}} 1/(X+\lambda)^n$.
- $G_n^{\Lambda_{\mathbf{z}}}(X)$ is a monic polynomial of degree n and $X \mid G_n^{\Lambda_{\mathbf{z}}}(X)$.

On the other hand, for any index $\mathfrak{a} = (a_1, \ldots, a_m)$, we define

$$\mathfrak{a}_{(i)} = (a_1, \ldots, a_i) \quad \text{and} \quad \mathfrak{a}^{(i)} = (a_{i+1}, \ldots, a_m).$$

We now introduce the notion of **multiple Goss sums (MGS) of rank r** together with the **Goss expansions** of MES.

Definition 2.8 (Chang–Chen–Huang–T., 2025)

Let $r \geq 2$ and $\mathfrak{a} = (a_1, \dots, a_m) \in \mathcal{I}$, we define

$$G_r(\mathfrak{a}; \mathbf{z}) = \sum_{\substack{f_1,\dots,f_m \in A_+ \\ \deg f_1 > \cdots > \deg f_m \geq 0}} G_{a_1}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(f_1 z_1)) \cdots G_{a_m}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(f_m z_1)).$$

Proposition 2.9 (Chang–Chen–Huang–T., 2025, 2026)

Let $r \geq 2$ and $\mathfrak{a} = (a_1, \dots, a_m) \in \mathcal{I}$. Then we have

$$\begin{aligned} E_r(\mathfrak{a}; \mathbf{z}) &= E_{r-1}(\mathfrak{a}; \mathbf{z}') + \sum_{i=1}^{m} E_{r-1}(\mathfrak{a}^{(i)}; \mathbf{z}') G_r(\mathfrak{a}_{(i)}; \mathbf{z}) \\ &= E_{r-1}(\mathfrak{a}; \mathbf{z}') + \sum_{n=1}^{\infty} A_n(\mathbf{z}') t_{\Lambda_{\mathbf{z}'}}(z_1)^n. \end{aligned}$$

Here is our first main result.

Theorem I (Chang–Chen–Huang–T., 2025)

For $r \geq 1$, we define $\widehat{E}_r : \mathcal{R} \to \mathcal{O}(\Omega^r)$ to be the unique $\mathbb{F}_p$-linear map satisfying

$$\widehat{E}_r(1) = 1 \quad \text{and} \quad \widehat{E}_r(x_{\mathfrak{a}}) = E_r(\mathfrak{a}; \mathbf{z}).$$

Then $\widehat{E}_r$ is an $\mathbb{F}_p$-algebra homomorphism. That is,

$$E_r(\mathfrak{a}; \mathbf{z}) \cdot E_r(\mathfrak{b}; \mathbf{z}) = \widehat{E}_r(x_{\mathfrak{a}}) \cdot \widehat{E}_r(x_{\mathfrak{b}}) = \widehat{E}_r(x_{\mathfrak{a}} * x_{\mathfrak{b}}).$$

Definition 3.1

Let $L \subseteq \mathbb{C}_\infty$ be an $\mathbb{F}_p$-subalgebra. For $r \geq 1$, we let

$$\mathcal{Z}_L^{(r)} = \operatorname{span}_L \{E_r(\mathfrak{a}; \mathbf{z}) \mid \mathfrak{a} \in \mathcal{I}\}.$$

As a corollary, we see that $\mathcal{Z}_L^{(r)}$ forms an L-algebra.

We briefly sketch the proof of Theorem I. The key idea is to use induction on r, taking Shi's result ($r = 1$) as the base case.
We first introduce the algebra setup for MES and MGS.

Definition 3.2

Define the q-shuffle algebra $(\mathcal{E}, *)$ as follows: Let S be the free monoid generated by $\{x_a, y_b \mid a, b \in \mathbb{N}\}$ subject to the relation $x_a y_b = y_b x_a$. Then

$$\mathcal{E} = \operatorname{span}_{\mathbb{F}_p} S = \operatorname{span}_{\mathbb{F}_p}\{x_{\mathfrak{a}} y_{\mathfrak{b}} \mid \mathfrak{a}, \mathfrak{b} \in \mathcal{I}\}.$$

The q-shuffle product is defined by:

- For all $X \in \mathcal{E}$,
$$1 * X = X * 1 = X.$$
- For all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$,
$$x_{\mathfrak{a}} * y_{\mathfrak{b}} = x_{\mathfrak{a}} y_{\mathfrak{b}} = y_{\mathfrak{b}} x_{\mathfrak{a}} = y_{\mathfrak{b}} * x_{\mathfrak{a}}.$$

Definition 3.2 (continued)

- For all $\mathfrak{a}, \mathfrak{b}$, $x_{\mathfrak{a}} * x_{\mathfrak{b}}$ is defined by the same formula as in $(R, *)$.
- For all $\mathfrak{a} = (a_1, \mathfrak{a}^{(1)}), \mathfrak{b} = (b_1, \mathfrak{b}^{(1)}) \in \mathcal{I}$,

$$y_{\mathfrak{a}} * y_{\mathfrak{b}} = y_{a_1}(y_{\mathfrak{a}^{(1)}} * y_{\mathfrak{b}}) + y_{b_1}(y_{\mathfrak{a}} * y_{\mathfrak{b}^{(1)}}) + y_{a_1+b_1}(y_{\mathfrak{a}^{(1)}} * y_{\mathfrak{b}^{(1)}})$$
$$+ \sum_{\substack{i+j=a_1+b_1 \\ (q-1)|j}} \Delta^{i,j}_{a_1,b_1} \left(y_i((y_{\mathfrak{a}^{(1)}} * y_{\mathfrak{b}^{(1)}}) * x_j) + y_i((y_{\mathfrak{a}^{(1)}} * y_{\mathfrak{b}^{(1)}}) * y_j)\right).$$

- For all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$,

$$(x_{\mathfrak{a}} y_{\mathfrak{b}}) * (x_{\mathfrak{a}'} y_{\mathfrak{b}'}) = (x_{\mathfrak{a}} * x_{\mathfrak{a}'}) * (y_{\mathfrak{b}} * y_{\mathfrak{b}'}).$$

For any $r \geq 2$, $y_{\mathfrak{b}}$ corresponds to the MGS $G_r(\mathfrak{b}; \mathbf{z})$ of rank r, and $x_{\mathfrak{a}}$ corresponds to the MES $E_{r-1}(\mathfrak{a}; \mathbf{z}')$ of rank $r-1$.

Then we have the following theorem:

Theorem 3.3 (Chang–Chen–Huang–T., 2025)

Let $r \geq 2$ and assume that Theorem I holds for rank $r-1$. Let $\widehat{G}_r : \mathcal{E} \to \mathcal{O}(\Omega^{r-1})$ be the unique $\mathbb{F}_p$-linear map such that

$$\widehat{G}_r(x_{\mathfrak{a}} y_{\mathfrak{b}}) = E_{r-1}(\mathfrak{a}; \mathbf{z}') G_r(\mathfrak{b}; \mathbf{z})$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$. Then $\widehat{G}_r$ is an $\mathbb{F}_p$-algebra homomorphism.

Thus, under the induction hypothesis, the q-shuffle algebra $(\mathcal{E}, *)$ provides an algebraic model for the q-shuffle product of MGS of rank r and MES of rank $r-1$. It remains to use the Goss expansions of MES to reconstruct the MES of rank r–1 and the MGS of rank r.

Hence, we define the map:

Definition 3.4

We define $\widehat{e} : \mathcal{R} \to \mathcal{E}$ to be the unique $\mathbb{F}_p$-linear map such that for any non-empty index $\mathfrak{a}$,

$$\widehat{e}(1) = 1 \quad \text{and} \quad \widehat{e}(x_{\mathfrak{a}}) = x_{\mathfrak{a}} + \sum_{i=1}^{\mathrm{dep}(\mathfrak{a})} x_{\mathfrak{a}^{(i)}} y_{\mathfrak{a}_{(i)}}.$$

We notice that by the Goss expansions of MES, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "\widehat{E}_r"] \arrow[d, "\widehat{e}"'] & \mathcal{O}(\Omega^r) \\
\mathcal{E} \arrow[ur, "\widehat{G}_r"'] &
\end{tikzcd}$$

Moreover, we see that the map factors through the quotient $\mathcal{E}/\mathfrak{A}$:

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\widehat{E}_r"] \arrow[d, "\widehat{e}"'] & \mathcal{O}(\Omega^r) \\ \mathcal{E}/\mathfrak{A} \arrow[ur, "\widehat{G}_r"'] & \end{tikzcd}$$

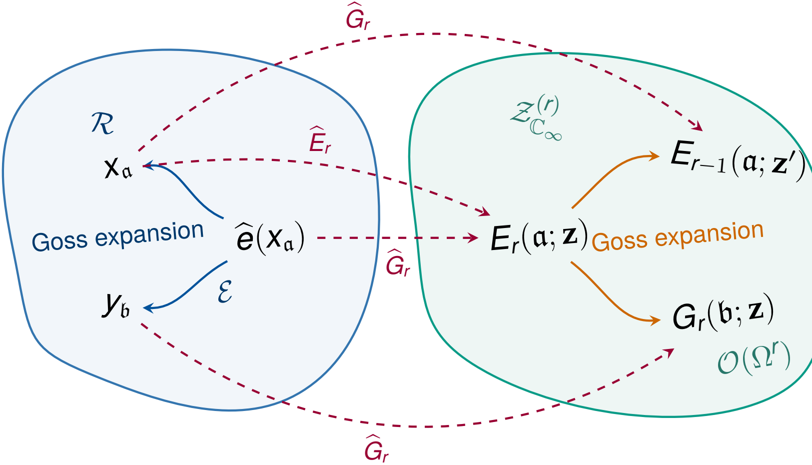
where $\mathfrak{A}$ is the associator ideal of $\mathcal{E}$. It remains to verify the following statement, which can be proved by utilizing associativity to reduce the computation.

Theorem 3.5 (Chang–Chen–Huang–T., 2026)

The map $\widehat{e} : \mathcal{R} \to \mathcal{E}/\mathfrak{A}$ is an $\mathbb{F}_p$-algebra homomorphism.

Therefore, Theorem I follows by induction, together with Theorems 3.3 and 3.5, and Shi's result (base case).

Correspondence of Objects



Theorem II (Chang–Chen–Huang–T., 2026)

The realization map $\widehat{E}_r : \mathcal{R} \to \mathcal{Z}_{\mathbb{F}_p}^{(r)}$ induces an injective L-algebra homomorphism

$$\widehat{E}_L : L \otimes_{\mathbb{F}_p} \mathcal{R} \hookrightarrow \varprojlim_{r\in\mathbb{N}} \mathcal{Z}_L^{(r)}, \quad c \otimes x_{\mathfrak{a}} \mapsto (cE_r(\mathfrak{a};\mathbf{z}))_{r\in\mathbb{N}},$$

with respect to the inverse system $\pi_{r+1,L} : \mathcal{Z}_L^{(r+1)} \to \mathcal{Z}_L^{(r)}$ where $\pi_{r+1,L}$ is the map taking the constant term.

Corollary (Chang–Chen–Huang–T., 2026)

We have the following:

- $(\mathcal{R}, *)$ is a commutative associative $\mathbb{F}_p$-algebra.
- The map $\widehat{e} : \mathcal{R} \to \mathcal{E}$ is an $\mathbb{F}_p$-algebra homomorphism.

Remark 4.1

The associativity of $(\mathcal{R}, *)$ was also proved by Im–Kim–Le–NgoDac–Pham (2023) using partial fractional decomposition and Thakur's power sums.

Finally, Theorem III provides the connection between the two q-shuffle algebras $(\mathcal{R}, *)$ and $(\mathcal{E}, *)$.

Theorem III (Chang–Chen–Huang–T., 2026)

Let $\phi : \mathcal{R} \otimes_{\mathbb{F}_p} \mathcal{R} \to \mathcal{E}$ be the unique $\mathbb{F}_p$-linear map such that

$$\phi(x_{\mathfrak{a}} \otimes x_{\mathfrak{b}}) = x_{\mathfrak{a}} * \widehat{e}(x_{\mathfrak{b}})$$

Then ϕ is an $\mathbb{F}_p$-algebra isomorphism. In particular, $(\mathcal{E}, *)$ is a commutative associative $\mathbb{F}_p$-algebra.

With Theorem III, one can verify the following:

Corollary (Chang–Chen–Huang–T., 2026)

For any Hopf $\mathbb{F}_p$-algebra structure on $(\mathcal{R}, *)$, there exists a unique Hopf $\mathbb{F}_p$-structure on $(\mathcal{E}, *)$ such that $\iota, \widehat{e} : \mathcal{R} \hookrightarrow \mathcal{E}$ are Hopf $\mathbb{F}_p$-algebra homomorphism. In this case, we have

$$\operatorname{Spec} \mathcal{E} \cong \operatorname{Spec} \mathcal{R} \times_{\mathbb{F}_p} \operatorname{Spec} \mathcal{R}$$

is an isomorphism as $\mathbb{F}_p$-group scheme.

Remark 4.2

A Hopf $\mathbb{F}_p$-algebra structure on $(\mathcal{R}, *)$ was conjectured by Shi (2018) and was announced by Im–Kim–Le–NgoDac–Pham (2023).

We now give a brief sketch of the proof of Theorem II. In fact, Theorem II follows immediately from the following key proposition:

Proposition 4.3 (Chang–Chen–Huang–T., 2026)

Let $w \in \mathbb{Z}_{\geq 0}$. Then for $r \gg 1$,

$$\{E_r(\mathfrak{a};\mathbf{z}) \mid \mathfrak{a} \in \mathcal{I}, \mathrm{wt}(\mathfrak{a}) \leq w\}$$

is a $\mathbb{C}_\infty$-linearly independent subset of $\mathcal{O}(\Omega^r)$.

To prove this, we consider the t-expansion. First, let

$$\mathrm{ord}_r(F) = \inf\{n \in \mathbb{Z} : F_n(\mathbf{z}') \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\}.$$

for any $F(\mathbf{z}) \in \mathcal{O}(\Omega^r)$ with expansion $F(\mathbf{z}) = \sum_{n\in\mathbb{Z}} F_n(\mathbf{z}')t_{\Lambda_{\mathbf{z}'}}(z_1)^n$.

By explicit computation, one checks the following lemma:

Lemma 4.4 (Chang–Chen–Huang–T., 2026)

Let $r \geq 2$. For any index $\mathfrak{a} \in \mathcal{I}$, we have

- *For any $a \in \mathbb{N}$ and $f \in A_+$,* $\operatorname{ord}_r\left(G_a^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(fz_1))\right) \geq q^{(r-1)\deg f}$.
- $\operatorname{ord}_r(G_r(\mathfrak{a};\mathbf{z})) \geq \frac{q^{(r-1)\operatorname{dep}(\mathfrak{a})}-1}{q^{r-1}-1}$.

Then we obtain

$$\begin{aligned} E_r(\mathfrak{a};\mathbf{z}) &= E_{r-1}(\mathfrak{a};\mathbf{z}') + E_{r-1}(\mathfrak{a}^{(1)};\mathbf{z}')G_r(\mathfrak{a}_{(1)};\mathbf{z}) \\ &\qquad + \sum_{i=2}^{\operatorname{dep}(\mathfrak{a})} E_{r-1}(\mathfrak{a}^{(i)};\mathbf{z}')G_r(\mathfrak{a}_{(i)};\mathbf{z}) \\ &= E_{r-1}(\mathfrak{a};\mathbf{z}') + E_{r-1}(\mathfrak{a}^{(1)};\mathbf{z}')G_r(\mathfrak{a}_{(1)};\mathbf{z}) + \bigstar \cdot t_{\Lambda_{\mathbf{z}'}}(z_1)^{q^{r-1}}. \end{aligned}$$

Moreover, write $\mathfrak{a} = (a_1, \mathfrak{a}^{(1)})$. Note that

$$
\begin{aligned}
G_r(\mathfrak{a}_{(1)}; \mathbf{z}) &= \sum_{f \in A_+} G_{a_1}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(fz_1)) \\
&= G_{a_1}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(z_1)) + \sum_{1 \neq f \in A_+} G_{a_1}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(fz_1)) \\
&= \text{lower degree terms} + t_{\Lambda_{\mathbf{z}'}}(z_1)^{a_1} + \bigstar \cdot t_{\Lambda_{\mathbf{z}'}}(z_1)^{q^{r-1}}.
\end{aligned}
$$

Therefore, we see that for $r \gg 1$, we have

$$
E_r(\mathfrak{a}; \mathbf{z}) = \text{lower degree terms} + E_{r-1}(\mathfrak{a}^{(1)}; \mathbf{z}') t_{\Lambda_{\mathbf{z}'}}(z_1)^{a_1} + \bigstar \cdot t_{\Lambda_{\mathbf{z}'}}(z_1)^{q^{r-1}}.
$$

Now, we are ready to prove the key proposition.

We illustrate the key idea by an example. Let $w = 8$. Assume that

$$\Xi = c_1 E_r(3,5;\mathbf{z}) + c_2 E_r(3,3,2;\mathbf{z}) + c_3 E_r(7,1;\mathbf{z}) = 0.$$

For $r \gg 1$, we have

$$\Xi = \text{lower degree terms} + c_3 E_{r-1}(1;\mathbf{z}') t_{\Lambda_{\mathbf{z}'}}(z_1)^7 + \bigstar \cdot t_{\Lambda_{\mathbf{z}'}}(z_1)^{q^{r-1}} = 0.$$

The uniqueness of t-expansion implies that

$$c_3 E_{r-1}(1;\mathbf{z}') = 0 \Rightarrow c_3 = 0.$$

Thus, we obtain

$$\Xi = c_1 E_r(3,5;\mathbf{z}) + c_2 E_r(3,3,2;\mathbf{z}) = 0.$$

For $r \gg 1$, we have

$$\Xi = \text{lower degree terms} + (c_1 E_{r-1}(5;\mathbf{z}') + c_2 E_{r-1}(3,2;\mathbf{z}'))t_{\Lambda_{\mathbf{z}'}}(z_1)^3 + \bigstar \cdot t_{\Lambda_{\mathbf{z}'}}(z_1)^{q^{r-1}} = 0.$$

The uniqueness of t-expansion implies that

$$\Xi' = c_1 E_{r-1}(5;\mathbf{z}') + c_2 E_{r-1}(3,2;\mathbf{z}') = 0.$$

The same argument shows that

$$c_1 E_{r-2}(\varnothing;\mathbf{z}'') = 0 \Rightarrow c_1 = 0.$$

Therefore, it follows that $c_1 = c_2 = c_3 = 0$, as desired.

It is natural to ask several questions about MES in this setting:

Question

- When does MES become a Drinfeld modular form? (partial result, Chen–Huang–T. (2026+))
- (analogue of the Goncharov direct sum conjecture) Is the algebra of MES graded by weight? (The case $r = 1$ is known by Chang (2014).) That is, do we have

$$\mathcal{Z}_{\overline{K}}^{(r)} = \bigoplus_{w=0}^{\infty} \mathcal{Z}_{w,\overline{K}}^{(r)},$$

where

$$\mathcal{Z}_{w,\overline{K}}^{(r)} = \mathrm{Span}_{\overline{K}}\{E_r(\mathfrak{a};\mathbf{z}) \mid \mathfrak{a} \in \mathcal{I},\ \mathrm{wt}(\mathfrak{a}) = w\}.$$

Question (continued)

- (analogue of the Zagier dimension conjecture and Hoffman's basis conjecture) What are the basis and dimension of $\mathcal{Z}^{(r)}_{w,\overline{K}}$? (The case $r = 1$ is known independently by Chang–Chen–Mishiba (2023) and Im–Kim–Le–NgoDac–Pham (2024).)
- Can we find a "fundamental" relation $\mathscr{R}$ such that all $\overline{K}$-linear relations among MES are "generated" by $\mathscr{R}$? (For $r = 1$, it is known that all relations among MZVs are generated by Thakur's fundamental relation $\mathscr{R}_1 : \zeta_A(q) - (\theta - \theta^q)\zeta_A(1, q-1)$.)
- More questions......

Thank You for Listening

ありがとうございました。