On Finite Multiple Harmonic u-Series over Function Field

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Outline

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Let

$$\mathcal{I} = \bigcup_{r=1}^{\infty} \mathbb{Z}_{\geq 1}^{r} \cup \{\varnothing\}$$

be the set of indices and

$$\mathcal{I}^{\mathrm{ad}} = \{(s_1, \ldots, s_r) \in \mathcal{I} : s_1 \geq 2\} \cup \{\varnothing\}$$

be the set of *admissible* indices. We recall the *multiple zeta values*.

Definition 1.1 (MZVs)

Let $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}^{\mathrm{ad}}$. Define

$$\zeta(\mathfrak{s}) = \sum_{m_1 > \cdots > m_r \geq 1} \frac{1}{m_1^{s_1} \cdots m_r^{s_r}} \in \mathbb{R}.$$

Denote by $\mathcal{Z} = \langle \zeta(\mathfrak{s}) : \mathfrak{s} \in \mathcal{I}^{\mathrm{ad}} \rangle_{\mathbb{Q}}$ the $\mathbb{Q}$-algebra generated by all MZVs.

Consider the algebra

$$\mathcal{A} = \frac{\prod_p \mathbb{F}_p}{\bigoplus_p \mathbb{F}_p}$$

where p runs over all prime numbers. Note that there is a natural embedding $\mathbb{Q} \hookrightarrow \mathcal{A}$. Then we define the *finite multiple zeta values*.

Definition 1.2 (FMZVs, Kaneko–Zagier)

Let $\mathbf{s} = (s_1, \ldots, s_r) \in \mathcal{I}$. Define

$$\zeta_{\mathcal{A}}(\mathbf{s}) = \left(\sum_{p > m_1 > \cdots > m_r \geq 1} \frac{1}{m_1^{s_1} \cdots m_r^{s_r}} \mod p \right)_p \in \mathcal{A}.$$

Denote by $\mathcal{Z}_{\mathcal{A}} = \langle \zeta(\mathbf{s}) : \mathbf{s} \in \mathcal{I} \rangle_{\mathbb{Q}}$ the $\mathbb{Q}$-algebra generated by all FMZVs.

We now recall the *symmetric multiple zeta values*. Following the framework of Ihara–Kaneko–Zagier (2006), we have the *stuffle-regularized MZVs*

$$\zeta_*(\mathbf{s};T) \in \mathbb{R}[T], \quad \zeta_*(\mathbf{s};T) = \zeta(\mathbf{s}) \text{ for } \mathbf{s} \in \mathcal{I}^{\mathrm{ad}}, \quad \zeta_*(1;T) = T.$$

Definition 1.3 (SMZVs, Kaneko–Zagier)

Let $\mathbf{s} = (s_1, \ldots, s_r) \in \mathcal{I}$. Define

$$\zeta_{\mathcal{S}}(\mathbf{s}) = \sum_{j=0}^{r} (-1)^{s_1+\cdots+s_j} \zeta_*(s_j, \ldots, s_1; T)\zeta_*(s_{j+1}, \ldots, s_r; T) \pmod{\zeta(2)\mathcal{Z}}.$$

We remark that this expression is independent of T and hence lies in $\mathcal{Z}$.

The Philosophy of Bachmann–Takeyama–Tasaka

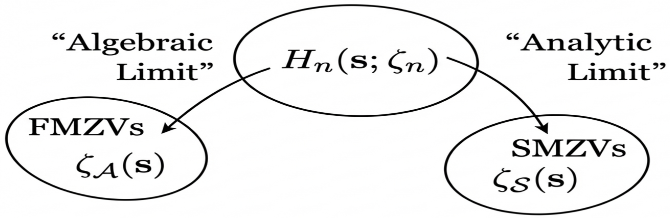
We introduce the philosophy of Bachmann–Takeyama–Tasaka, originating from their work in 2018: Let $\mathrm{s} = (s_1, \dots, s_r) \in \mathcal{I}$. Consider the *finite multiple harmonic q-series*

$$H_n(\mathrm{s}; q) = \sum_{n > m_1 > \cdots > m_r > 0} \frac{q^{(s_1 - 1)m_1 + \cdots + (s_r - 1)m_r}}{[m_1]_q^{s_1} \cdots [m_r]_q^{s_r}},$$

where

$$[m]_q := \frac{q^m - 1}{q - 1}$$

denotes the q-integer. Let $\zeta_n = \exp(2\pi\sqrt{-1}/n)$. They show that the "limit" of finite multiple harmonic q-series at roots of unity are related to both FMZVs and SMZVs. Their work can be illustrated by the following figure:



Theorem 1.4 (Bachmann–Takeyama–Tasaka, 2018)

$$\zeta_{\mathcal{A}}(\mathrm{s}) = \left(H_p(\mathrm{s};\zeta_p) \mod (1-\zeta_p)\right)_p.$$

Theorem 1.5 (Bachmann–Takeyama–Tasaka, 2018)

$$\zeta_{\mathcal{S}}(\mathrm{s}) \equiv \Re\left(\lim_{n\to\infty} H_n(\mathrm{s};\zeta_n)\right) \pmod{\zeta(2)\mathcal{Z}}.$$

This serves as evidence for the Kaneko–Zagier conjecture, as both FMZVs and SMZVs arise from the same object.

Kaneko–Zagier Conjecture

There is a $\mathbb{Q}$-algebra isomorphism $\varphi_{\mathrm{KZ}} : \mathcal{Z}_{\mathcal{A}} \to \mathcal{Z}/\zeta(2)\mathcal{Z}$ such that

$$\varphi_{\mathrm{KZ}}(\zeta_{\mathcal{A}}(\mathbf{s})) = \zeta_{\mathcal{S}}(\mathbf{s})$$

for all $\mathbf{s} \in \mathcal{I}$.

The Kaneko–Zagier conjecture predicts that FMZVs and SMZVs satisfy the same algebraic relations. Indeed, they share many properties; for example, both vanish at depth one.

Function Field Setting

Let p be a prime and let q be a power of p. The function field setting is given as follows:

$$
\begin{tikzcd}
A := \mathbb{F}_q[\theta] \arrow[d, leftrightarrow] & \subseteq & K := \mathbb{F}_q(\theta) \arrow[d, leftrightarrow] & \subseteq & K_\infty := \mathbb{F}_q((1/\theta)) \arrow[d, leftrightarrow] & \subseteq & \mathbb{C}_\infty := \widehat{\overline{K_\infty}} \arrow[d, leftrightarrow] \\
\mathbb{Z} & \subseteq & \mathbb{Q} & \subseteq & \mathbb{R} & \subseteq & \mathbb{C}
\end{tikzcd}
$$

We let $A_+ \subseteq A$ be the set of monic polynomials, which plays the role of the positive integers $\mathbb{N}$. Here, we also normalize the ∞-adic absolute value on K such that $|\theta|_\infty = q$.

We recall the *Carlitz module*, defined as the functor

$$\mathbf{C} : \{A\text{-algebras}\} \to \{A\text{-modules}\}$$

which assigns to an A-algebra B the A-module structure on $\mathbf{C}(B) = B$ via the homomorphism

$$C : A \to \mathrm{End}_{\mathbb{F}_q}(B), \quad \theta \mapsto \theta \,\mathrm{id} + \tau,$$

where $\tau(b) = b^q$ is the Frobenius endomorphism. Notice that $\mathbf{C}$ serves as the function field analogue of the multiplicative group

$$\mathbb{G}_m : \{\mathbb{Z}\text{-algebras}\} \to \{\mathbb{Z}\text{-modules}\}.$$

The analogy is also manifested by the following correspondences:

1. Exponential Short Exact Sequences

- ($\mathbb{Z}$-modules):

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathrm{Lie}(\mathbb{G}_m)(\mathbb{C}) = \mathbb{C} \xrightarrow{\exp} \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times \longrightarrow 1.$$

- (A-modules):

$$0 \longrightarrow \tilde{\pi}A \longrightarrow \mathrm{Lie}(\mathbf{C})(\mathbb{C}_\infty) = \mathbb{C}_\infty \xrightarrow{\exp_{\mathbf{C}}} \mathbf{C}(\mathbb{C}_\infty) = \mathbb{C}_\infty \longrightarrow 0.$$

Here, $\tilde{\pi}$ is the *Carlitz period* and $\exp_C$ is the *Carlitz exponential*.

2. Cyclic Torsion Modules

- Roots of unity: $\mathbb{G}_m(\mathbb{C})[n]$ is generated by $\zeta_n = \exp(2\pi\sqrt{-1}/n)$ for $n \in \mathbb{N}$.
- Carlitz torsion points: $\mathbf{C}(\mathbb{C}_\infty)[\mathfrak{n}]$ is generated by $\lambda_\mathfrak{n} = \exp_{\mathbf{C}}(\tilde{\pi}/\mathfrak{n})$ for $\mathfrak{n} \in A_+$.

Now, we introduce the MZVs in positive characteristic.

Definition 2.1 (Thakur's MZVs, Thakur, 2004)

Let $\mathbf{s} = (s_1, \ldots, s_r) \in \mathcal{I}$. Define

$$\zeta_A(\mathbf{s}) = \sum_{\substack{a_1,\ldots,a_r \in A_+ \\ \deg a_1 > \cdots > \deg a_r \geq 0}} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in K_\infty.$$

Denote by $\mathcal{Z}_\infty = \langle \zeta_A(\mathbf{s}) : \mathbf{s} \in \mathcal{I} \rangle_K$ the K-algebra generated by all Thakur's MZVs.

In 2009, Thakur proved the existence of the *q-shuffle relations*. That is, for any indices $\mathbf{s}, \mathbf{t} \in \mathcal{I}$, the product $\zeta_A(\mathbf{s})\zeta_A(\mathbf{t})$ can be written as an $\mathbb{F}_p$-linear combination of Thakur's MZVs of the same weight $\operatorname{wt} \mathbf{s} + \operatorname{wt} \mathbf{t}$.

Example 2.2 (Chen, 2015)

$$\zeta_A(s)\zeta_A(t) = \zeta_A(s,t) + \zeta_A(t,s) + \zeta_A(s,t) + \sum_{\substack{i+j=s+t \\ (q-1)|j}} \Delta_{s,t}^{i,j}\zeta_A(i,j)$$

where $s,t \in \mathbb{N}$ and $\Delta_{s,t}^{i,j} := (-1)^{s-1}\binom{j-1}{s-1} + (-1)^{t-1}\binom{j-1}{t-1} \in \mathbb{F}_p$.

We now introduce the *q-shuffle algebra*

$$\mathcal{R} = \mathrm{span}_{\mathbb{F}_p}\{x_{\mathbf{s}} : \mathbf{s} \in \mathcal{I}\}$$

whose product $*$ was first observed by Yamamoto based on Chen's formula, and later explicitly formulated in Shi's 2018 PhD thesis. Through this algebraic structure, the q-shuffle relations are captured by the following theorem:

Theorem 2.3 (Shi, 2018)

The realization map

$$\widehat{\zeta_A} : \mathcal{R} \to K_\infty, \quad x_{\mathbf{s}} \mapsto \zeta_A(\mathbf{s})$$

is an $\mathbb{F}_p$-algebra homomorphism. That is, $\widehat{\zeta_A}(x_{\mathbf{s}} * x_{\mathbf{t}}) = \zeta(\mathbf{s})\zeta(\mathbf{t})$ for all $\mathbf{s}, \mathbf{t} \in \mathcal{I}$.

Next, we introduce the *FMZVs over K*. Consider the algebra

$$\mathcal{A}_K = \frac{\prod_v \mathbb{F}_v}{\bigoplus_v \mathbb{F}_v}$$

where v runs over all monic irreducible polynomials in A and $\mathbb{F}_v = A/(v)$ denotes the residue field. Then we have a natural embedding $K \hookrightarrow \mathcal{A}_K$.

Definition 2.4 (FMZVs over K, Chang–Mishiba, 2017)

Let $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$. Define

$$\zeta_{\mathcal{A}_K}(\mathfrak{s}) = \left(\sum_{\substack{a_1,\ldots,a_r \in A_+ \\ \deg v > \deg a_1 > \cdots > \deg a_r \geq 0}} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \mod v \right)_v \in \mathcal{A}_K.$$

Denote by $\mathcal{Z}_{\mathcal{A}_K} = \langle \zeta(\mathfrak{s}) : \mathfrak{s} \in \mathcal{I} \rangle_K$ the K-algebra generated by all FMZVs over K.

Note that FMZVs over K were also investigated in Shi's PhD thesis.

Definition 3.1

For $a \in A$ and $u \in \mathbb{C}_\infty$, define

$$[a]_u = \begin{cases} \frac{C_a(u)}{u}, & \text{if } u \neq 0, \\ a, & \text{if } u = 0. \end{cases}$$

One checks immediately that as $u \to 0$, $[a]_u \to a$. Then we have the notion of *finite multiple harmonic u-series*:

Definition 3.2

$$Z_n(\mathrm{s}; u) := \sum_{\substack{a_1, \ldots, a_m \in A_+ \\ n > \deg a_1 > \cdots > \deg a_m \geq 0}} \frac{1}{[a_1]_u^{s_1} \cdots [a_m]_u^{s_m}}$$

for $\mathrm{s} \in \mathcal{I}$ and $u \in \mathbb{C}_\infty$ with $C_a(u) \neq 0$ for all $a \in A_{+,\deg < n}$.

One sees that $\lim_{n\to\infty}\lim_{u\to 0} Z_n(\mathbf{s}; u) = \zeta_A(\mathbf{s})$. Then we have the following theorems regarding the finite multiple harmonic u-series at Carlitz torsion points:

Theorem 3.3 (T., 2026+)

For all $\mathbf{s} \in \mathcal{I}$, *we have*

$$\zeta_{\mathcal{A}_K}(\mathbf{s}) = \left(Z_{\deg v}(\mathbf{s}; \lambda_v) \mod \lambda_v\right)_v .$$

Theorem 3.4 (T., 2026+)

For all $\mathbf{s} \in \mathcal{I}$, *we have*

$$\zeta_A(\mathbf{s}) = \lim_{\deg \mathfrak{n}\to\infty} Z_{\deg \mathfrak{n}}(\mathbf{s}; \lambda_{\mathfrak{n}}).$$

These two theorems serve as the function field analogs of the results established by Bachmann–Takeyama–Tasaka in 2018.

Theorem 3.5 (Finite Euler–Carlitz Formula, T., 2026+)

Let $s \in \mathbb{N}$ be q-even, i.e., $(q-1) \mid s$ and $\mathfrak{n} \in A_+$. We have

$$Z_{\deg \mathfrak{n}}(s; \lambda_{\mathfrak{n}}) = \frac{\mathrm{dBC}_s(\mathfrak{n})}{\Gamma_{s+1}} \cdot (\mathfrak{n}\lambda_{\mathfrak{n}})^s \in K \cdot (\mathfrak{n}\lambda_{\mathfrak{n}})^s.$$

Here, $\mathrm{dBC}_s(\mathfrak{n})$ denotes the degenerate Bernoulli-Carlitz number, and Γ_{s+1} is the Carlitz factorial.

Together with Theorems 3.3 and 3.4, we obtain:

Corollary 3.6

Let $s \in \mathbb{N}$ be q-even, i.e., $(q-1) \mid s$.

$$\zeta_{\mathcal{A}_K}(s) = (0)_v.$$

We mention that Shi proved Corollary 3.6 via a completely different approach in her PhD thesis (2018).

Corollary 3.7 (Euler–Carlitz Formula)

Let $s \in \mathbb{N}$ be q-even, i.e., $(q-1) \mid s$. We have

$$\zeta_A(s) = \frac{\mathrm{BC_s}}{\Gamma_{s+1}} \cdot \tilde{\pi}^s.$$

Next, we let $\mathfrak{D}$ be the set consisting of $u \in \mathbb{C}_\infty$ satisfying one of the following conditions:

- $u = 0$.
- $|u|_\infty > q^{1/(q-1)}$.
- $|u|_\infty < q^{1/(q-1)}$ and $|u| \notin q^{\mathbb{Z}_{\leq 0}+1/(q-1)}$.

Definition 3.8

For $\mathfrak{s} \in \mathcal{I}$, define the *u-multiple zeta values* by

$$\zeta_u(\mathfrak{s}) = \lim_{n\to\infty} Z_{<n}(\mathfrak{s}; u) = \sum_{\substack{a_1,\ldots,a_m \in A_+ \\ \deg a_1 > \cdots > \deg a_m \geq 0}} \frac{1}{[a_1]_u^{s_1} \cdots [a_m]_u^{s_m}},$$

which converges for any $u \in \mathfrak{D}$.

In general, $\lim_{u\to 0} \zeta_u(\mathfrak{s})$ may not exist. But we can find a sequence of u_m such that $\lim_{m\to\infty} \zeta_{u_m}(\mathfrak{s}) = \zeta_A(\mathfrak{s})$.

Remark 3.9

The u-analogs of arithmetic, geometric, and two-variable gamma functions are similarly well-defined and well-behaved on $\mathfrak{D}$.

We now discuss the q-shuffle relations.

Theorem 3.10 (T., 2026+)

The realization map

$$\widehat{Z_n(\bullet; u)} : \mathcal{R} \to \mathbb{C}_\infty, \quad x_{\mathrm{s}} \mapsto Z_n(\mathrm{s}; u)$$

is an $\mathbb{F}_p$-algebra homomorphism.

As a consequence, we obtain:

Corollary 3.11

Thakur's MZVs, FMZVs over K and u-MZVs all satisfy the q-shuffle relations.

Indeed, Thakur's MZVs and FMZVs over K both satisfy the q-shuffle relations, as they are defined via multiple power sums.

Based on B-T-T philosophy, it is natural to consider the Kaneko–Zagier type conjecture in our setting. Based on some computational results presented in Shi's PhD thesis, we make the following conjecture.

Conjecture 1

There exists a well-defined surjective K-algebra homomorphism

$$\varphi : \mathcal{Z}_{\mathcal{A}_K} \longrightarrow \mathcal{Z}_\infty / \zeta_A(q-1)\mathcal{Z}_\infty$$

satisfying

$$\varphi\big(\zeta_{\mathcal{A}_K}(\mathbf{s})\big) = \zeta_A(\mathbf{s}) \pmod{\zeta_A(q-1)}$$

for any indices $\mathbf{s} \in \mathcal{I}$, and $\ker\varphi$ is the ideal generated by all binary but not fixed relations of multiple zeta values.

Conjecture 2

$\ker\varphi$ is the ideal generated by the fundamental relation

$$\ker\phi = \langle \zeta_{\mathcal{A}_K}(q) + (\theta^q - \theta)\zeta_{\mathcal{A}_K}(1, q-1)\rangle\,.$$

We briefly explain the binary relations and fixed relations. For $\mathrm{s} \in \mathcal{I}$ and $d \in \mathbb{Z}$, define the *multiple power sum*

$$S_d(\mathrm{s}) := \sum_{\substack{a_1,\dots,a_r \in A_+ \\ d = \deg a_1 > \cdots > \deg a_r}} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}}$$

where the empty sum is defined to be zero. We see that

$$\zeta_A(\mathrm{s}) = \sum_{d\in\mathbb{Z}} S_d(\mathrm{s}).$$

Fixed and Binary Relations

Definition 4.1

Let $\sum_i r_i \zeta_A(\mathrm{s}_i) = 0$ be a K-linear relation. This relation is called *binary* if there exist a_i, b_i such that $a_i + b_i = r_i$ and

$$\sum_i a_i S_d(\mathrm{s}_i) + \sum_i b_i S_{d+1}(\mathrm{s}_i) = 0$$

for all $d \in \mathbb{Z}$, and it is called *fixed* if $b_i = 0$ for all i.

One checks that FMZVs over K satisfy all the fixed relations. In fact, it is known that every K-linear relation among Thakur's MZVs comes from binary relations. Relations among Thakur's MZVs are well-understood. See Todd (2018), Ngo Dac (2021), Chang–Chen–Mishiba (2022), and Im–Kim–Le–Ngo Dac–Pham (2023).

Drinfeld Generalization

A *Drinfeld module* of rank r is defined via the functor

$$\mathbf{E} : \{A\text{-algebras}\} \to \{A\text{-modules}\}$$

where the A-module structure on $\mathbf{E}(B) = B$ is given by

$$\phi : A \to \mathrm{End}_{\mathbb{F}_q}(B), \quad \theta \mapsto \theta \cdot \mathrm{id} + g_1\tau + \cdots + g_r\tau^r, \quad g_r \neq 0.$$

It can be viewed as the function field analogs of elliptic curves (rank-two object) with arbitrary rank. Fix any Drinfeld module $\mathbf{E} = (\mathbb{G}_a, \phi)$. One can naturally define the $(u, \mathbf{E})$-bracket by

$$[a]_{u,\mathbf{E}} = \phi_a(u)/u.$$

It is interesting to consider the generalization of the results above. Theorem 3.10 still holds and Theorem 3.5 admits a generalization. However, other properties stay unknown.

Thank You for Listening

ありがとうございました。