On q-shuffle Relations for Multiple Eisenstein Series in Positive Characteristic

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Japan-Taiwan Number Theory Conference

September 3, 2025

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Classical MZVs

Definition

Let $\underline{s} = (s_1, \ldots, s_\ell) \in \mathbb{N}^\ell$ with $s_1 > 1$. The *multiple zeta value* (abbreviated as *MZV*), is defined by

$$\zeta(\underline{s}) = \zeta(s_1, \ldots, s_\ell) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \cdots n_\ell^{s_\ell}} \in \mathbb{R}.$$

The quantities $\mathrm{wt}(\underline{s}) := \sum_{i=1}^{\ell} s_i$ and $\mathrm{dep}(\underline{s}) := \ell$ are called the *weight* and *depth* of the presentation $\zeta(\underline{s})$, respectively.

Integral Representations of MZVs

Playing around with iterated integrals, the MZV $\zeta(s_1,\ldots,s_\ell)$, $s_1>1$ admits the $\mathrm{wt}(\underline{s})$-dimensional iterated integral representation

$$\zeta(s_1,\ldots,s_\ell)=\int_0^1\left(\omega_0^{s_1-1}\omega_1\right)\left(\omega_0^{s_2-1}\omega_1\right)\cdots\left(\omega_0^{s_\ell-1}\omega_1\right)$$

where we denote $\omega_0=dx/x$ and $\omega_1=dx/(1-x)$. For example, we have

$$\zeta(2,3)=\int_0^1\frac{dx_1}{x_1}\int_0^{x_1}\frac{dx_2}{1-x_2}\int_0^{x_2}\frac{dx_3}{x_3}\int_0^{x_3}\frac{dx_4}{x_4}\int_0^{x_4}\frac{dx_5}{1-x_5}.$$

Stuffle Relations

There are many relations among MZVs; for example, Euler's non-trivial identity $\zeta(2,1)=\zeta(3)$. In this context, we introduce the *stuffle* and *shuffle relations*.

Example (Stuffle Relations)

For $s,t>1$, we have

$$\zeta(s)\zeta(t)=\sum_{n,m\geq 1}\frac{1}{n^s m^t}=\left(\sum_{n>m\geq 1}+\sum_{m>n\geq 1}+\sum_{m=n\geq 1}\right)\frac{1}{n^s m^t}$$
$$=\zeta(s,t)+\zeta(t,s)+\zeta(s+t).$$

Shuffle Relations

Example (Shuffle Relations)

We have

$$\zeta(2)\zeta(2) = \left(\int_{1>x_1\geq x_2>0} \frac{dx_1}{x_1}\frac{dx_2}{1-x_2}\right)\left(\int_{1>y_1\geq y_2>0} \frac{dy_1}{y_1}\frac{dy_2}{1-y_2}\right)$$
$$= \int_{\substack{1>x_1\geq x_2>0\\ 1>y_1\geq y_2>0}} \frac{dx_1}{x_1}\frac{dx_2}{1-x_2}\frac{dy_1}{y_1}\frac{dy_2}{1-y_2}.$$

Decomposing the domain of integration, this becomes

$$4\zeta(3,1)+2\zeta(2,2).$$

Classical MES

Let $\tau \in \mathbb{H}$. We define a partial order $\succ$ on the lattice $\mathbb{Z}\tau + \mathbb{Z}$ as follows:

1. $m\tau + n \succ 0$ if $m > 0$ or $(m, n) \in \{0\} \times \mathbb{N}$.
2. $m_1\tau + n_1 \succ m_2\tau + n_2$ if $(m_1 - n_1)\tau + (m_2 - n_2) \succ 0$.

Definition (Gangl–Kaneko–Zagier 2006, Bachmann 2012)

For $\underline{s} = (s_1, \ldots, s_\ell) \in \mathbb{N}^\ell$ with $s_1 \geq 3$ and $s_2, \ldots, s_\ell \geq 2$ and $\tau \in \mathbb{H}$, the *multiple Eisenstein series* (abbreviated as *MES*), is defined by

$$G_{\underline{s}}(\tau) := \sum_{\substack{m_1\tau+n_1,\ldots,m_\ell\tau+n_\ell\in\mathbb{Z}\tau+\mathbb{Z} \\ m_1\tau+n_1\succ\cdots\succ m_\ell\tau+n_\ell\succ 0}} \frac{1}{(m_1\tau + n_1)^{s_1} \cdots (m_\ell\tau + n_\ell)^{s_\ell}}.$$

Fourier Expansion of MES

The Fourier expansion of multiple Eisenstein series is given by the following theorem.

Theorem (Gangl–Kaneko–Zagier 2006, Bachmann 2012)

For $\underline{s} = (s_1, \ldots, s_\ell) \in \mathbb{N}^\ell$ with $s_1 \geq 3$ and $s_2, \ldots, s_\ell \geq 2$, we have

$$G_{\underline{s}}(\tau) = \zeta(s_1, \ldots, s_\ell) + \sum_{n=1}^{\infty} a_n q^n$$

where $q = e^{2\pi i \tau}$ and a_n can be explicitly written down in terms of MZVs and generalized divisor functions.

Since MZVs appear in the constant terms of MES, it is natural to ask whether relations among MZVs can be lifted to relations among the corresponding MES. In particular, one may ask whether MES satisfy the stuffle and shuffle relations.

$$
\begin{tikzcd}
\textit{MES} \arrow[d, "\textit{const.term}"] & \textit{Relations among MES} \\
\textit{MZVs} & \textit{Relations among MZVs} \arrow[u, dashed, "\textit{lifting?}"]
\end{tikzcd}
$$

Question: Do the stuffle and shuffle relations among MZVs lift to the corresponding MES?

Lifting Relations from MZVs to MES

However, some shuffle and stuffle relations cannot be lifted directly because the corresponding multiple Eisenstein series may not be defined due to convergence issues. For example, in the shuffle relation

$$\zeta(3)\zeta(3) = 12\zeta(5,1) + 6\zeta(4,2) + 2\zeta(3,3),$$

the Eisenstein series for $\underline{s} = (5,1)$ is not defined.

Theorem (Bachmann–Tasaka 2018, Bachmann 2019)

The shuffle (resp. stuffle) regularized multiple Eisenstein series satisfy the same shuffle (resp. stuffle) relations as the corresponding multiple zeta values.

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Fix a prime p and let q be a power of p. We consider the function field settings:

Classical		**Function Field**
$\mathbb{N}$	$\longleftrightarrow$	$A_+ = \{f \in A \mid f \text{ is monic}\}$
$\mathbb{Z}$	$\longleftrightarrow$	$A = \mathbb{F}_q[\theta]$
$\mathbb{Q}$	$\longleftrightarrow$	$K = \mathbb{F}_q(\theta)$
$\mathbb{R}$	$\longleftrightarrow$	$K_\infty = \mathbb{F}_q((1/\theta))$
$\mathbb{C}$	$\longleftrightarrow$	$\mathbb{C}_\infty = \widehat{\overline{K_\infty}}$

MZVs in Positive Characteristic

Definition (Thakur 2004)

Let $\underline{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$. *Thakur's multiple zeta value* is defined by

$$\zeta_A(\underline{a}) := \sum_{\substack{f_1,\ldots,f_m \in A_+ \\ \deg f_1 > \cdots > \deg f_m \geq 0}} \frac{1}{f_1^{a_1} \cdots f_m^{a_m}} \in K_\infty.$$

Here, we recall that the classical MZV is defined by

$$\zeta(\underline{s}) = \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \cdots n_\ell^{s_\ell}} \in \mathbb{R}.$$

Power Sums

Definition (Thakur 2004)

Let $\underline{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$ and $d \in \mathbb{Z}_{\geq 0}$. We define the *power sum* by

$$S_d(\underline{a}) := \sum_{\substack{f_1,\ldots,f_m \in A_+ \\ d = \deg f_1 > \cdots > \deg f_m \geq 0}} \frac{1}{f_1^{a_1} \cdots f_m^{a_m}} \in K.$$

Using the notion of power sums, Thakur's multiple zeta value can be expressed as

$$\zeta(\underline{a}) = \sum_{d=0}^{\infty} S_d(\underline{a}).$$

q-shuffle Relations

From now on, the term "MZVs" refers to Thakur's multiple zeta values.
Thakur showed that the product of two MZVs can be expressed as an $\mathbb{F}_p$-linear combination of MZVs of the same weight. That is, for any indices $\underline{a}$ and $\underline{b}$, we have

$$\zeta_A(\underline{a})\zeta_A(\underline{b}) = \mathbb{F}_p\text{-linear combinations of } \zeta(\underline{c}_i)\text{'s} \qquad (*_q)$$

where each $\underline{c}_i$ is an index with $\mathrm{wt}(\underline{c}_i) = \mathrm{wt}\,(\underline{a}) + \mathrm{wt}\,(\underline{b})$. These relation among MZVs are called *q-shuffle relations*. The first concrete example is given by Chen in 2015.

Example (Chen 2015)

For any $a, b \in \mathbb{N}$,

$$\zeta_A(a)\zeta_A(b) = \zeta_A(a,b) + \zeta_A(b,a) + \zeta_A(a+b) + \sum_{\substack{i+j=a+b \\ q-1|j}} \Delta_{a,b}^{i,j} \zeta_A(i,j)$$

where $\Delta_{a,b}^{i,j} := (-1)^{a-1}\binom{j-1}{a-1} + (-1)^{b-1}\binom{j-1}{b-1} \in \mathbb{F}_p$.

We compare this with the classical stuffle and shuffle relations:

$$\zeta(s)\zeta(t) = \zeta(s,t) + \zeta(t,s) + \zeta(s+t) \quad \text{(stuffle)}$$

$$= \sum_{j=2}^{s+t-1} \left(\binom{j-1}{s-1} + \binom{j-1}{t-1} \right) \zeta(j, s+t-j). \quad \text{(shuffle)}$$

Question: How can one give an explicit expression of these q-shuffle relations $(*_q)$ for higher depth cases?

Based on Chen's formula for the depth-one case, Yamamoto observed that these q-shuffle relations are expected to satisfy explicit inductive formulas, and this observation leads to the following approach.

Answer: The composition space $\mathcal{R}$.

- $\mathcal{R}$ consists of $\mathbb{F}_p$-linear combinations of formal words $x_{\underline{a}}$.
- $\mathcal{R}$ is an $\mathbb{F}_p$-algebra, with a product that encodes the expected q-shuffle relations, defined via the inductive formulas.

With this algebraic setup, giving an explicit expression of $(*_q)$ reduces to saying that the realization map $x_{\underline{a}} \mapsto \zeta_A(\underline{a})$ is an $\mathbb{F}_p$-algebra homomorphism (proved by Shi).

The Composition Space $\mathcal{R}$

We now present a formal definition of $\mathcal{R}$. Let $\mathcal{S}$ be the free monoid generated by the set $\{x_n \mid n \in \mathbb{N}\}$ and $\mathcal{R}$ be the $\mathbb{F}_p$-vector space generated by $\mathcal{S}$.
For $\underline{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$, we denote $\underline{a}^- = (a_2, \ldots, a_m)$.
Moreover, let $x_{\underline{a}} = x_{a_1} \cdots x_{a_m}$ and $x_\varnothing = 1$. Such an element is called a *word*. We make $\mathcal{R}$ into an $\mathbb{F}_p$-algebra as follows:

- For the empty word $x_\varnothing = 1$ and any $\mathfrak{a} \in \mathcal{R}$, we define

$$1 * \mathfrak{a} = \mathfrak{a} * 1 = \mathfrak{a}.$$

The Composition Space $\mathcal{R}$

- (Yamamoto) For $\underline{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$ and $\underline{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$, we define

$$\begin{aligned} x_{\underline{a}} * x_{\underline{b}} = x_{a_1}(x_{\underline{a}^-} * x_{\underline{b}}) + x_{b_1}(x_{\underline{a}} * x_{\underline{b}^-}) + x_{a_1+b_1}(x_{\underline{a}^-} * x_{\underline{b}^-}) \\ + \sum_{\substack{i+j=a_1+b_1 \\ q-1 \mid j}} \Delta^{i,j}_{a_1,b_1} x_i((x_{\underline{a}^-} * x_{\underline{b}^-}) * x_j). \end{aligned}$$

Remark

A similar algebraic setup for the stuffle and shuffle relations among classical MZVs was introduced by Hoffman.

With the composition space $\mathcal{R}$ in hand, we consider the realization maps. For $d \in \mathbb{N}$, let $\widehat{S}_{<d}, \widehat{\zeta}_A : \mathcal{R} \to K_\infty$ be the unique $\mathbb{F}_p$-linear maps satisfying

$$\widehat{S}_{<d}(x_{\underline{a}}) := S_{<d}(\underline{a}) := \sum_{i=0}^{d-1} S_i(\underline{a}), \quad \widehat{\zeta}_A(x_{\underline{a}}) := \lim_{d\to\infty} \widehat{S}_{<d}(x_{\underline{a}}) = \zeta_A(\underline{a}).$$

Theorem (Shi 2018)

For any $d \in \mathbb{N}$, $\widehat{S}_{<d}$ is an $\mathbb{F}_p$-algebra homomorphism and hence so is $\widehat{\zeta}_A$. That is,

$$\widehat{\zeta}_A(x_{\underline{a}} * x_{\underline{b}}) = \zeta_A(\underline{a})\zeta_A(\underline{b}).$$

Drinfeld Symmetric Space

Let $r \geq 1$. Let Ω^r be the *Drinfeld symmetric space* of rank r consisting of elements in $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$ with K_∞-linearly independent entries. We viewed Ω^r as a subset of $\mathbb{C}_\infty^r$ by normalizing the last coordinate to 1.

$\mathbb{H}$	$\longleftrightarrow$	Ω^r
$\tau = [\tau : 1] \in \mathbb{H}$	$\longleftrightarrow$	$\mathbf{z} = [z_1 : \cdots : z_{r-1} : 1] \in \Omega^r$
$\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$	$\longleftrightarrow$	$\Lambda_{\mathbf{z}} = Az_1 + \cdots + Az_{r-1} + A$

To define the MES in positive characteristic, we first introduce a partial order on $\Lambda_{\mathbf{z}} = Az_1 + \cdots + Az_{r-1} + A$.

Partial Order on Lattices

For $\boldsymbol{f} = (f_1, \ldots, f_r) \in A^r$ and $\mathbf{z} = (z_1, \ldots, z_r) \in \Omega^r$, we put

$$\langle \boldsymbol{f}, \mathbf{z} \rangle := \sum_{i=1}^{r} f_i z_i \in \Lambda_{\mathbf{z}} = Az_1 + \cdots + Az_{r-1} + A.$$

We define a partial order $\succ$ on $\Lambda_{\mathbf{z}}$ as follows: Let $\langle \boldsymbol{f}, \mathbf{z} \rangle$ and $\langle \boldsymbol{g}, \mathbf{z} \rangle \in \Lambda_{\mathbf{z}}$.

- $\langle \boldsymbol{f}, \mathbf{z} \rangle \succ 0$ if $f_1 = \cdots = f_{i-1} = 0$ and $f_i \in A_+$ for some $1 \leq i \leq r$.
- For $\langle \boldsymbol{f}, \mathbf{z} \rangle, \langle \boldsymbol{g}, \mathbf{z} \rangle \succ 0$, we write $\langle \boldsymbol{f}, \mathbf{z} \rangle \succ \langle \boldsymbol{g}, \mathbf{z} \rangle$ if one of the following holds:

Partial Order on Lattices

- There exists $1 \leq i \leq r$ such that $f_1 = \cdots = f_{i-1} = g_1 = \cdots = g_{i-1} = 0$ and $f_i, g_i \in A_+$ with $\deg f_i > \deg g_i$.
- There exists $1 \leq i \leq r$ such that $f_1 = \cdots = f_{i-1} = g_1 = \cdots = g_{i-1} = 0$ and $f_i \in A_+, g_i = 0$.

For $r = 2$, this partial order was introduced by Chen. With this notion in place, we are now ready to define the multiple Eisenstein series.

MES in Positive Characteristic

Definition (Chen 2017, Chang–Chen–Huang–T. 2025)

Let $r \geq 1$. For any non-empty index $\underline{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$, we define the *multiple Eisenstein series of rank* r on Ω^r by

$$E_r(\underline{a}; \mathbf{z}) := \sum_{\substack{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m \in A^r \\ \langle \boldsymbol{f}_1, \mathbf{z} \rangle \succ \cdots \succ \langle \boldsymbol{f}_m, \mathbf{z} \rangle \succ 0}} \frac{1}{\langle \boldsymbol{f}_1, \mathbf{z} \rangle^{a_1} \cdots \langle \boldsymbol{f}_m, \mathbf{z} \rangle^{a_m}}.$$

By convention, we put $E_r(\varnothing; \mathbf{z}) := 1$.

We remark that when $r = 1$, $E_1(\underline{a}; \mathbf{z}) = \zeta_A(\underline{a})$.

Question: Do the q-shuffle relations among MZVs lift to the corresponding MES?

$$\begin{tikzcd} \textit{MES} \arrow[d, "\text{``const.term``}"] & \textit{Relations among MES} \\ \textit{MZVs} & \textit{Relations among MZVs} \arrow[u, dashed, "\textit{lifting?}"'] \end{tikzcd}$$

Partial Results: In 2017, Chen proved that the q-shuffle relations for products of single zeta values (depth one) can be lifted to multiple Eisenstein series of rank 2.

Main Theorem

For $r \geq 1$, we define $\widehat{E}_r : \mathcal{R} \to \mathcal{O}(\Omega^r)$ to be the unique $\mathbb{F}_p$-linear map satisfying

$$\widehat{E}_r(1) := 1 \quad \text{and} \quad \widehat{E}_r(x_{\underline{a}}) := E_r(\underline{a}; \mathbf{z}).$$

Theorem (Chang–Chen–Huang–T. 2025)

For $r \geq 1$, $\widehat{E}_r$ is an $\mathbb{F}_p$-algebra homomorphism. That is,

$$\widehat{E}_r(x_{\underline{a}} * x_{\underline{b}}) = E_r(\underline{a}; \mathbf{z}) E_r(\underline{b}; \mathbf{z}).$$

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Some Notations

We first define the following notations:

Definition

1. For $\underline{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$ and $0 \leq i \leq m-1$, we define
$$\underline{a}^{(i)} := (a_{i+1}, \ldots, a_m), \quad \underline{a}^{(m)} := \varnothing,$$
$$\underline{a}_{(i)} := (a_1, \ldots, a_i) \quad \text{and} \quad \underline{a}_{(0)} := \varnothing.$$
2. For $r \geq 2$ and $\mathbf{z} = (z_1, \ldots, z_{r-1}, z_r = 1) \in \Omega^r$, let
$$\mathbf{z}' := (z_2, \ldots, z_r) \in \Omega^{r-1}.$$

Goss Polynomials

For $r \geq 1$ and $\mathbf{z} = (z_1, \ldots, z_r) \in \Omega^r$ with the associated A-lattice $\Lambda_{\mathbf{z}} := Az_1 + \cdots + Az_r$ in $\mathbb{C}_\infty$, we consider the function

$$t_{\Lambda_{\mathbf{z}}}(x) := \sum_{\lambda \in \Lambda_{\mathbf{z}}} \frac{1}{x + \lambda}.$$

In 1980, Goss introduced a sequence of polynomials $G_k^{\Lambda_z}(x) \in \mathbb{C}_\infty[x]$ satisfying

$$G_k^{\Lambda_{\mathbf{z}}}(t_{\Lambda_{\mathbf{z}}}(x)) = \sum_{\lambda \in \Lambda_{\mathbf{z}}} \frac{1}{(x + \lambda)^k}.$$

With these notions at hand, we can define the Goss power sums and multiple Goss sums.

Goss Power Sums and Multiple Goss Sums

Definition

For $\underline{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$ and $d \in \mathbb{Z}_{\geq 0}$, we define the *Goss power sum* on Ω^r by

$$G_r^{=d}(\underline{a}; \mathbf{z}) := \sum_{\substack{f_1,\ldots,f_m \in A_+ \\ d = \deg f_1 > \cdots > \deg f_m \geq 0}} G_{a_1}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(f_1 z_1)) \cdots G_{a_m}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(f_m z_1)).$$

We define the *multiple Goss sum* on Ω^r by

$$G_r(\underline{a}; \mathbf{z}) := \lim_{d \to \infty} G_r^{<d}(\underline{a}; \mathbf{z}) := \lim_{d \to \infty} \sum_{i=0}^{d-1} G_r^{=i}(\underline{a}; \mathbf{z}).$$

Expansion of MES

Theorem (Chang–Chen–Huang–T. 2025)

Let $r \geq 2$. For $\underline{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$ and $\mathbf{z} \in \Omega^r$, we have

$$E_r(\underline{a}; \mathbf{z}) = E_{r-1}(\underline{a}; \mathbf{z}') + \sum_{i=1}^{m} E_{r-1}(\underline{a}^{(i)}; \mathbf{z}')G_r(\underline{a}_{(i)}; \mathbf{z}).$$

Note that for $r = 2$, the MZV occurs in the "constant term" of the expansion. This expansion enables us to prove the main theorem by induction, with the MZV serving as the base case.

Strategy of Proof

By direct computation and observation, we formulate the q-shuffle product for multiple Goss sums. Our strategy is then to define a new q-shuffle algebra $\mathcal{E}$, which consists of $\mathbb{F}_p$-linear combinations of formal words $x_{\underline{a}}y_{\underline{b}}$. The product of the $x_{\underline{a}}$ encodes the q-shuffle product of MZVs, while the product of the $y_{\underline{b}}$ encodes the q-shuffle product of multiple Goss sums. With the algebra at hand, we decompose the map

$$\widehat{E}_r : \mathcal{R} \to \mathcal{O}(\Omega^r), \quad x_{\underline{a}} \mapsto E_r(\underline{a}; \mathbf{z})$$

into

$$\widehat{E}_r = \widehat{G}_r \circ \widehat{e} : \mathcal{R} \to \mathcal{E}/\mathfrak{A} \to \mathcal{O}(\Omega^r).$$

Strategy of Proof

Here, $\mathfrak{A}$ denotes the associator ideal of $\mathcal{E}$, $\widehat{e} : \mathcal{R} \to \mathcal{E}/\mathfrak{A}$ is the formalization of the expansion of MES, defined by

$$\widehat{e}(x_{\underline{a}}) := x_{\underline{a}} + \sum_{i=1}^{\mathrm{dep}(\underline{a})} x_{\underline{a}^{(i)}} y_{\underline{a}_{(i)}} \pmod{\mathfrak{A}},$$

and $\widehat{G}_r : \mathcal{E}/\mathfrak{A} \to \mathcal{O}(\Omega^r)$ is the realization map of rank r multiple Goss sums, defined by

$$\widehat{G}_r(x_{\underline{a}} y_{\underline{b}}) = E_{r-1}(a; \mathbf{z}') G_r(\underline{b}; z).$$

Therefore, our main theorem follows from verifying that both $\widehat{e}$ and $\widehat{G}_r$ are $\mathbb{F}_p$-algebra homomorphisms under induction hypothesis.

Thank you for your attention.