

ON MULTIPLE EISENSTEIN SERIES IN POSITIVE CHARACTERISTIC: DIRECT SUM RESULT

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ABSTRACT. In this paper, we study multiple Eisenstein series (MES) in positive characteristic. By computing and analyzing the t -expansions of MES, we determine the precise “weights” of their coefficients. This framework enables us to establish a graded algebra structure for MES, extending the direct sum result for Thakur’s multiple zeta values proved in [Cha14]. Our result may also be viewed as a function field analogue of the corresponding direct sum result of Bachmann and Kanno [BK26].

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1. INTRODUCTION

1.1. Classical story. Throughout this paper, we denote by $\mathbb{Z}$ the set of integers. An index refers to the empty index $\emptyset$ or a tuple of positive integers. For any nonempty index

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$\mathfrak{s} = (s_1, \dots, s_m)$, we put

$$\text{wt}(\mathfrak{s}) = s_1 + \dots + s_m \quad \text{and} \quad \text{dep}(\mathfrak{s}) = m,$$

which are called the weight and depth of the index $\mathfrak{s}$, respectively. By convention, we define $\text{wt}(\emptyset) = \text{dep}(\emptyset) = 0$.

Classical real multiple zeta values (MZVs for short) are generalizations of the Riemann zeta values at positive integers, originating in the works of Euler, Hoffman, and Zagier [Hof92, Zag94]. They are defined, for any nonempty index $\mathfrak{s} = (s_1, \dots, s_m)$ with $s_1 \geq 2$, by the following series

$$\zeta(s_1, \dots, s_m) := \sum_{n_1 > \dots > n_m \geq 1} \frac{1}{n_1^{s_1} \dots n_m^{s_m}}.$$

MZVs have been intensively studied since their introduction, as they occur in number theory and arithmetic geometry as well as their intersection (see [And04, Zha16, BF26]).

A remarkable deformation of MZVs is provided by the multiple Eisenstein series (abbreviated as MES) initiated by Gangl-Kaneko-Zagier [GKZ06] in the depth two case, and generalized by Bachmann for higher depth [Bac12]. We denote by $\mathbb{H}$ the complex upper half-plane. For any nonempty index $\mathfrak{s} = (s_1, \dots, s_m)$ with $s_1 \geq 2, \dots, s_m \geq 2$, the associated MES is the multiple lattice sum

$$G_{\mathfrak{s}}(\tau) := \sum_{\lambda_1 \succ \dots \succ \lambda_m \succ 0} \frac{1}{\lambda_1^{s_1} \dots \lambda_m^{s_m}}, \quad \tau \in \mathbb{H},$$

where the summation is taken over the lattice $\mathbb{Z}\tau + \mathbb{Z}$. When $s_1 = 2$, we use Eisenstein summation (see [Bac23]). Here, $\lambda = a\tau + b \succ 0$ means $a > 0$, or $a = 0$ and $b > 0$, and $\lambda \succ \lambda'$ means $\lambda - \lambda' \succ 0$. Based on [GKZ06, Bac12], the constant term of the Fourier expansion of $G_{\mathfrak{s}}(\tau)$ is the corresponding MZV $\zeta(\mathfrak{s})$.

For any integer $w \geq 2$, we denote by $\mathfrak{Z}_w$ (resp. $\mathcal{E}_w$) the $\mathbb{Q}$ -vector space spanned by $\zeta(\mathfrak{s})$ for $\mathfrak{s} = (s_1, \dots, s_m)$ with $s_1 \geq 2$ (resp. $G_{\mathfrak{s}}(\tau)$ for $\mathfrak{s} = (s_1, \dots, s_m)$ with $s_1 \geq 2, \dots, s_m \geq 2$) and $\text{wt}(\mathfrak{s}) = w$. We further set

$$\mathfrak{Z}_0 = \mathcal{E}_0 = \mathbb{Q}, \quad \mathfrak{Z}_1 = \mathcal{E}_1 = \{0\}.$$

Put $\mathfrak{Z} = \sum_{w \geq 0} \mathfrak{Z}_w$ and $\mathcal{E} := \sum_{w \geq 0} \mathcal{E}_w$. From the harmonic product, $\mathfrak{Z}$ and $\mathcal{E}$ form $\mathbb{Q}$ -algebras. More precisely, we have that for integers $w_1, w_2 \geq 0$,

$$\mathfrak{Z}_{w_1} \cdot \mathfrak{Z}_{w_2} \subseteq \mathfrak{Z}_{w_1+w_2}, \quad \mathcal{E}_{w_1} \cdot \mathcal{E}_{w_2} \subseteq \mathcal{E}_{w_1+w_2}.$$

The celebrated Goncharov–Zagier direct sum conjecture [Zag94, Gon01] predicts that there should be no nontrivial $\mathbb{Q}$ -linear relations among MZVs of different weights:

Conjecture 1.1 (Goncharov–Zagier). *The $\mathbb{Q}$ -algebra $\mathfrak{Z}$ is a graded $\mathbb{Q}$ -algebra, i.e., $\mathfrak{Z} = \bigoplus_{w \geq 0} \mathfrak{Z}_w$.*

In the very recent paper [BK26], Bachmann and Kanno give an affirmative answer to the analogue of Goncharov–Zagier’s conjecture for MES in the following results.

Theorem 1.2 (Bachmann–Kanno [BK26]). *Let $\mathcal{E}_{\mathbb{C}}$ be the $\mathbb{C}$ -vector space spanned by all multiple Eisenstein series. Then the following results hold.*

- (i) $\mathcal{E} = \bigoplus_{w \geq 0} \mathcal{E}_w$.
- (ii) The natural map $\mathcal{E} \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \mathcal{E}_{\mathbb{C}}$ is an isomorphism.

In the function field setting, the first named author [Cha14] proved an analogue of Goncharov–Zagier’s direct sum conjecture for Thakur’s MZVs by incorporating the Baker–Wüstholz–Yu philosophy [Wus89, BW07, Yu97]. The purpose of this paper is to extend the result of [Cha14] to MES in positive characteristic defined in [Che17, CCHT25].

1.2. Function field analogue. Throughout this paper, we fix the following notation. Let p be a prime and let q be a power of p . Let $\mathbb{F}_q$ denote the finite field with q elements and θ be a variable. We denote by $A := \mathbb{F}_q[\theta]$ the polynomial ring in θ over $\mathbb{F}_q$ with field of fractions $K := \mathbb{F}_q(\theta)$. Let $K_{\infty} := \mathbb{F}_q((1/\theta))$ be the completion of K at the infinite place ∞ , with uniformizer $1/\theta$, and let $\mathbb{C}_{\infty} := \widehat{\overline{K}_{\infty}}$ be the completion of a fixed algebraic closure of K_{∞} . We let $|\cdot|$ be the absolute value on $\mathbb{C}_{\infty}$ normalized such that $|\theta| = q$. We further denote by $\overline{K}$ the algebraic closure of K in $\mathbb{C}_{\infty}$. Finally, we put $A_+ = \{a \in A \mid a \text{ is monic}\}$.

For any nonempty index $\mathfrak{s} = (s_1, \dots, s_m)$, Thakur’s MZV at $\mathfrak{s}$ [Tha04] is defined by

$$\zeta_A(\mathfrak{s}) := \sum_{\substack{f_1, \dots, f_m \in A_+ \\ \deg f_1 > \dots > \deg f_m}} \frac{1}{f_1^{s_1} \cdots f_m^{s_m}} \in K_{\infty}.$$

Carlitz zeta values [Car35] are special cases of Thakur’s MZVs with depth one. By convention, we put $\zeta_A(\emptyset) = 1$.

For any subfield L of $\mathbb{C}_{\infty}$ containing K and integer $w \geq 0$, we put

$$\mathcal{Z}_{L,w} := \text{span}_L \{ \zeta_A(\mathfrak{s}) \mid \text{wt}(\mathfrak{s}) = w \}.$$

In [Tha10], Thakur discovered and proved the so-called q -shuffle product, which implies that for any integers $w_1, w_2 \geq 0$,

$$\mathcal{Z}_{L,w_1} \cdot \mathcal{Z}_{L,w_2} \subseteq \mathcal{Z}_{L,w_1+w_2}.$$

It follows that the L -vector space spanned by all Thakur’s MZVs, denoted by

$$\mathcal{Z}_L := \sum_{w=0}^{\infty} \mathcal{Z}_{L,w},$$

forms an L -algebra.

Using the linear independence criterion of Anderson–Brownawell–Papanikolas [ABP04] as well as period interpretation of Thakur’s MZVs proved by Anderson–Thakur [AT09], we have the following stronger form of function field analogue of Goncharov–Zagier’s direct sum conjecture.

Theorem 1.3 ([Cha14, Theorem 2.2.1]). *We keep the notation as above. Then the following results hold.*

- (i) $\mathcal{Z}_{\overline{K}}$ is a graded algebra, i.e., $\mathcal{Z}_{\overline{K}} = \bigoplus_{w=0}^{\infty} \mathcal{Z}_{\overline{K},w}$.
- (ii) The natural map $\mathcal{Z}_K \otimes_K \overline{K} \rightarrow \mathcal{Z}_{\overline{K}}$ is an isomorphism.

In other words, $\mathcal{Z}_{\bar{K}}$ is a graded algebra “defined over K ” in the sense that K -linear independence of Thakur’s MZVs implies $\bar{K}$ -linear independence.

Motivated by Gangl–Kaneko–Zagier, MES in positive characteristic were introduced by Chen [Che17] in the case of rank two and depth two, and the theory was later extended to MES of arbitrary rank and arbitrary depth in [CCHT25]. Hereafter, MES will always refer to multiple Eisenstein series in positive characteristic whenever it is clear from the context. The primary goal of this paper is to extend Theorem 1.3 to MES. We briefly recall their definition here, and refer to §2 for further details.

Fix a positive integer r . For any $\mathbf{f} = (f_1, \dots, f_r) \in A^r$ and $\mathbf{z} = (z_1, \dots, z_r) \in \mathbb{C}_\infty^r$, we put

$$\langle \mathbf{f}, \mathbf{z} \rangle := \mathbf{f} \mathbf{z}^{\text{tr}} = f_1 z_1 + \dots + f_r z_r \in \mathbb{C}_\infty.$$

By viewing the set A_+ as an analogue of the set of positive integers, we define a partial order $\succ$ on A^r given in Definition 2.1, and then define the following MES of rank r , which are rigid analytic functions on the Drinfeld symmetric space

$$\Omega^r = \{ \mathbf{z} = (z_1, \dots, z_r) \in \mathbb{C}_\infty^r \mid z_1, \dots, z_r \text{ are } K_\infty\text{-linearly independent and } z_r = 1 \}.$$

Definition 1.4 ([CCHT25, Definition 1.4]). Fix a positive integer r . For any $\mathbf{z} \in \Omega^r$ and nonempty index $\mathfrak{s} = (s_1, \dots, s_m)$, we define the multiple Eisenstein series of rank r by

$$E_r(\mathfrak{s}; \mathbf{z}) = \sum_{\substack{\mathbf{f}_1, \dots, \mathbf{f}_m \in A^r \\ \mathbf{f}_1 \succ \dots \succ \mathbf{f}_m \succ 0}} \frac{1}{\langle \mathbf{f}_1, \mathbf{z} \rangle^{s_1} \dots \langle \mathbf{f}_m, \mathbf{z} \rangle^{s_m}}.$$

We adopt the convention that $E_r(\emptyset; \mathbf{z}) := 1$.

Note that the above definition was initiated by Chen [Che17] in the case of $r = 2$ and $m = 2$. We also mention that, in [Pel25], Pellarin studied vector-valued versions of MES in the rank two case. In analogy with the classical case, our MES can be regarded as a higher-rank deformation of Thakur’s MZVs as we have

$$E_1(\mathfrak{s}; \mathbf{z}) = \zeta_A(\mathfrak{s})$$

(see [CCHT25, p. 4]). Moreover, for $r \geq 2$, the MES $E_r(\mathfrak{s}; \mathbf{z})$ admits a “Goss expansion” with constant term $E_{r-1}(\mathfrak{s}; \mathbf{w})$ (see Proposition 2.6).

Let $r \geq 1$. For any subfield L of $\mathbb{C}_\infty$ containing K and integer $w \geq 0$, we set

$$\mathcal{Z}_{L,w}^{(r)} := \text{span}_L \{ E_r(\mathfrak{s}; \mathbf{z}) \mid \text{wt}(\mathfrak{s}) = w \}$$

and let

$$\mathcal{Z}_L^{(r)} := \sum_{w=0}^{\infty} \mathcal{Z}_{L,w}^{(r)}.$$

It is shown in [CCHT25, Theorem 1.6] that Thakur’s q -shuffle product can be lifted to MES. Therefore, for any integers $w_1, w_2 \geq 0$,

$$(1.2.1) \quad \mathcal{Z}_{L,w_1}^{(r)} \cdot \mathcal{Z}_{L,w_2}^{(r)} \subseteq \mathcal{Z}_{L,w_1+w_2}^{(r)}$$

and $\mathcal{Z}_L^{(r)}$ becomes an L -algebra.

1.3. Main results and idea of proof. The main result of this paper is stated as follows.

Theorem 1.5. *For any integer $r \geq 1$, the following results hold.*

(i) $\mathcal{Z}_{\overline{K}}^{(r)}$ is a graded algebra, i.e.,

$$\mathcal{Z}_{\overline{K}}^{(r)} = \bigoplus_{w=0}^{\infty} \mathcal{Z}_{\overline{K},w}^{(r)}.$$

(ii) The natural map $\mathcal{Z}_K^{(r)} \otimes_K \overline{K} \twoheadrightarrow \mathcal{Z}_{\overline{K}}^{(r)}$ is an isomorphism of $\overline{K}$ -algebras.

The proof of our main results is outlined as follows. We proceed by induction on r to establish Theorem 1.5, with the base case $r = 1$ following from Theorem 1.3. The bridge from MES of rank $r + 1$ to MES of rank r is the “expansion at infinity” in the parameter $t_{\Lambda_{z'}}$. The key ingredient of the induction step is to analyze the $t_{\Lambda_{z'}}$ -expansion of any MES of rank $r + 1$. We compute its coefficients and show that each coefficient can be expressed in terms of MES of rank r , divided by a suitable power of the Drinfeld discriminant form Δ_r (see §3.1).

The explicit description mentioned above allows us to determine the “weights” of the coefficients appearing in the $t_{\Lambda_{z'}}$ -expansion. Consequently, any linear relation among MES of rank $r + 1$ induces, coefficient by coefficient, corresponding relations among MES of rank r . Using the uniqueness of the $t_{\Lambda_{z'}}$ -expansion together with the induction hypothesis for MES of rank r , we then obtain the desired conclusion for MES of rank $r + 1$, and hence establish Theorem 1.5 (see §3 for details).

Remark 1.6. After completing the proofs of our main results, we learned that Bachmann and Kanno had posted on arXiv a paper proving Theorem 1.2. We then observed that their method could be adapted to strengthen our Theorem 1.5 by replacing $\overline{K}$ with $\mathbb{C}_{\infty}$ when $r \geq 2$. Precisely, the proof of the first assertion in arbitrary rank and that of the second assertion in rank two follow a strategy similar to that of [BK26], while the second assertion in arbitrary rank is obtained by combining the rank two case with the induction argument used in the proof of Theorem 1.5. We refer to Appendix A for details.

2. PRELIMINARIES

In this section, we recall the necessary preliminaries on Drinfeld modular forms, multiple Eisenstein series, and multiple Goss sums, which will be used primarily in the proof of Theorem 1.5 in §3.

2.1. The Drinfeld symmetric space Ω^r . Given a positive integer $r \geq 1$, we identify the Drinfeld symmetric space Ω^r of rank r with the following subset of $\mathbb{C}_{\infty}^r$:

$$\Omega^r = \{\mathbf{z} = (z_1, \dots, z_r) \in \mathbb{C}_{\infty}^r \mid z_1, \dots, z_r \text{ are } K_{\infty}\text{-linearly independent and } z_r = 1\}.$$

It is known from [Dri74, Proposition 6.1] that Ω^r admits a rigid analytic structure and we denote by $\mathcal{O}(\Omega^r)$ the $\mathbb{C}_{\infty}$ -algebra of rigid analytic functions on Ω^r .

Then for any $\mathbf{z} = (z_1, \dots, z_{r-1}, 1) \in \Omega^r$ and $\mathbf{f} = (f_1, \dots, f_r) \in A^r$, we denote by $\langle \mathbf{f}, \mathbf{z} \rangle := f_1 z_1 + \dots + f_{r-1} z_{r-1} + f_r \in \mathbb{C}_\infty$. We put

$$\Lambda_{\mathbf{z}} := \{ \langle \mathbf{f}, \mathbf{z} \rangle \mid \mathbf{f} \in A^r \} \subseteq \mathbb{C}_\infty,$$

and note that $\Lambda_{\mathbf{z}}$ forms an A -lattice of rank r inside $\mathbb{C}_\infty$.

We now recall the notion of the $t_{\Lambda_{\mathbf{z}'}}$ -expansion. Assume $r \geq 2$. For $\mathbf{z} = (z_1, \dots, z_{r-1}, 1) \in \Omega^r$, we put $\mathbf{z}' = (z_2, \dots, z_{r-1}, 1) \in \Omega^{r-1}$ and consider the function

$$(2.1.1) \quad t_{\Lambda_{\mathbf{z}'}}(z_1) := \sum_{\lambda \in \Lambda_{\mathbf{z}'}} \frac{1}{z_1 + \lambda} = \exp_{\Lambda_{\mathbf{z}'}}(z_1)^{-1},$$

where

$$\exp_{\Lambda_{\mathbf{z}'}}(z_1) := z_1 \prod_{0 \neq \lambda \in \Lambda_{\mathbf{z}'}} \left(1 - \frac{z_1}{\lambda} \right)$$

is the exponential function associated with the rank $r-1$ lattice $\Lambda_{\mathbf{z}'}$ (see [Gos96, §4]).

Let

$$\Gamma_r := \left\{ \left(\frac{1}{0} \middle| \begin{smallmatrix} * \\ I_{r-1} \end{smallmatrix} \right) \right\} \subseteq \mathrm{GL}_r(A)$$

and denote the subspace of all Γ_r -invariant functions by $\mathcal{O}(\Omega^r)^{\Gamma_r}$. By [BBP24, Proposition 5.4] (see also [Gek25]), every $F \in \mathcal{O}(\Omega^r)^{\Gamma_r}$ admits a unique $t_{\Lambda_{\mathbf{z}'}}$ -expansion:

$$F(\mathbf{z}) = \sum_{n=-\infty}^{\infty} F_n(\mathbf{z}') t_{\Lambda_{\mathbf{z}'}}(z_1)^n,$$

valid for $\mathbf{z} = (z_1, \dots, z_{r-1}, 1) \in \Omega^r$ in a neighborhood of infinity. We further define the order of F at infinity by $\mathrm{ord}_r(F) := \inf\{n \in \mathbb{Z} \mid F_n(\mathbf{z}') \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\}$.

2.2. Multiple Eisenstein series and multiple Goss sums. To define MES, we first recall the partial order introduced in [Che17, CCHT25], reformulated here as a partial order on A^r .

Definition 2.1. For every nonzero $\mathbf{f} = (f_1, \dots, f_r) \in A^r$, we put

$$\ell(\mathbf{f}) := \min\{i \in \{1, \dots, r\} \mid f_i \neq 0\}.$$

We define a partial order $\succ$ on A^r as follows. For nonzero $\mathbf{f} \in A^r$, we write $\mathbf{f} \succ 0$ if $f_{\ell(\mathbf{f})} \in A_+$. For $\mathbf{f}, \mathbf{g} \succ 0$, we write $\mathbf{f} \succ \mathbf{g}$ if either $\ell(\mathbf{f}) < \ell(\mathbf{g})$, or

$$\ell(\mathbf{f}) = \ell(\mathbf{g}) \quad \text{and} \quad \deg f_{\ell(\mathbf{f})} > \deg g_{\ell(\mathbf{g})}.$$

For any $r \geq 1$ and any nonempty index $\mathfrak{s} = (s_1, \dots, s_m)$, recall that the associated MES of rank r is defined by

$$E_r(\mathfrak{s}; \mathbf{z}) = \sum_{\substack{\mathbf{f}_1, \dots, \mathbf{f}_m \in A^r \\ \mathbf{f}_1 \succ \dots \succ \mathbf{f}_m \succ 0}} \frac{1}{\langle \mathbf{f}_1, \mathbf{z} \rangle^{s_1} \dots \langle \mathbf{f}_m, \mathbf{z} \rangle^{s_m}}.$$

It was shown in [CCHT25] that $E_r(\mathfrak{s}; \mathbf{z})$ is a rigid analytic function on Ω^r . Moreover, for $r \geq 2$, it is invariant under Γ_r , and therefore admits a unique $t_{\Lambda_{\mathbf{z}'}}$ -expansion.

Given $r \geq 1$ and an A -lattice Λ of rank r in $\mathbb{C}_\infty$, we write

$$(2.2.1) \quad \exp_\Lambda(X) := X \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{X}{\lambda}\right) = \sum_{n=0}^{\infty} e_n(\Lambda) X^{q^n}.$$

For $\mathbf{z} \in \Omega^r$, we also put $e_n(\mathbf{z}) := e_n(\Lambda_{\mathbf{z}})$. By [BBP24, Proposition 15.3], the function $\mathbf{z} \mapsto e_n(\mathbf{z}) : \Omega^r \rightarrow \mathbb{C}_\infty$ is a rigid analytic function on Ω^r . Hence, we may regard $\exp_{\Lambda_{\mathbf{z}}}(X)$ as a power series with coefficients in $\mathcal{O}(\Omega^r)$.

With the necessary setup, we recall the notion of Goss polynomials:

Definition 2.2 (Goss polynomials, [Gos80, Gek88]). Fix $r \geq 1$ and an A -lattice Λ of rank r in $\mathbb{C}_\infty$. We define the Goss polynomials by the sequence of polynomials $\{G_k^\Lambda(X)\}_{k=1}^\infty$ in $\mathbb{C}_\infty[X]$ satisfying the following recurrence:

- (i) $G_1^\Lambda(X) = X$, and
- (ii) for an integer $k > 1$,

$$G_k^\Lambda(X) = X \sum_{q^i < k} e_i(\Lambda) G_{k-q^i}^\Lambda(X).$$

Since $\mathbf{z} \mapsto e_n(\mathbf{z})$ is a rigid analytic function on Ω^r , the coefficients of $G_k^{\Lambda_{\mathbf{z}}}(X)$ are also rigid analytic functions on Ω^r . In addition, if $\exp_\Lambda(X) \in K[[X]]$, then $G_k^\Lambda(X) \in K[X]$ for every integer $k \geq 1$.

The core property of Goss polynomials is that they express the reciprocal power sums over lattices in $\mathbb{C}_\infty$ as polynomials in $t_{\Lambda_{\mathbf{z}}}(X)$, where $t_{\Lambda_{\mathbf{z}}}(X)$ is obtained from (2.1.1) by substituting $\mathbf{z}' \rightarrow \mathbf{z}$ and $z_1 \rightarrow X$.

Proposition 2.3 ([Gos80, Gek88]). *For any integer $k \geq 1$, the following properties hold.*

- (i) $G_k^{\Lambda_{\mathbf{z}}}(t_{\Lambda_{\mathbf{z}}}(X)) = \sum_{\lambda \in \Lambda_{\mathbf{z}}} \frac{1}{(X+\lambda)^k}$.
- (ii) $G_k^{\Lambda_{\mathbf{z}}}(X)$ is a monic polynomial of degree k .
- (iii) $X \mid G_k^{\Lambda_{\mathbf{z}}}(X)$.
- (iv) if $1 \leq k \leq q$, $G_k^{\Lambda_{\mathbf{z}}}(X) = X^k$.

Using Goss polynomials, we have the notion of multiple Goss sums (abbreviated as MGS) introduced in [CCHT25].

Definition 2.4 (Multiple Goss sums). For any $r \geq 2$ and nonempty index $\mathfrak{s} = (s_1, \dots, s_m)$, we define the multiple Goss sum of rank r by

$$G_r(\mathfrak{s}; \mathbf{z}) = \sum_{\substack{f_1, \dots, f_m \in A_+ \\ \deg f_1 > \dots > \deg f_m}} G_{s_1}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(f_1 z_1)) \cdots G_{s_m}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(f_m z_1)).$$

By convention, we put $G_r(\emptyset; \mathbf{z}) = 1$.

We also recall the notations $\mathfrak{s}^{(i)}$ and $\mathfrak{s}_{(i)}$. The former one is the index obtained from $\mathfrak{s}$ by deleting the first i entries and the latter one is the index obtained from $\mathfrak{s}$ by retaining the first i entries. The precise definitions are given as follows.

Definition 2.5. For a nonempty index $\mathfrak{s} = (s_1, \dots, s_m)$ and $i \geq 0$, we let

$$\mathfrak{s}^{(i)} = \begin{cases} \mathfrak{s}, & i = 0, \\ (s_{i+1}, \dots, s_m), & 1 \leq i < m, \\ \emptyset, & i \geq m, \end{cases} \quad \text{and} \quad \mathfrak{s}_{(i)} = \begin{cases} \emptyset, & i = 0, \\ (s_1, \dots, s_i), & 1 \leq i < m, \\ \mathfrak{s}, & i \geq m. \end{cases}$$

Using Proposition 2.3 and Definition 1.4, we obtain the Goss expansion of MES:

Proposition 2.6 ([CCHT25, Proposition 2.3]). *For any integer $r \geq 2$ and any index $\mathfrak{s}$, we have*

$$E_r(\mathfrak{s}; \mathbf{z}) = E_{r-1}(\mathfrak{s}; \mathbf{z}') + \sum_{i=1}^{\text{dep}(\mathfrak{s})} E_{r-1}(\mathfrak{s}^{(i)}; \mathbf{z}') G_r(\mathfrak{s}_{(i)}; \mathbf{z}).$$

The authors of [CCHT26] computed the $t_{\Lambda_{\mathbf{z}'}}$ -expansion of $t_{\Lambda_{\mathbf{z}'}}(f_1 z_1)$ and established the following lower bound (see the proof of [CCHT26, Lemma 2.9]):

$$(2.2.2) \quad \text{ord}_r(t_{\Lambda_{\mathbf{z}'}}(f_1 z_1)) \geq q^{(r-1) \deg f}.$$

Furthermore, by using (2.2.2), it is shown in [CCHT26, Proposition 2.10] that the constant term of the $t_{\Lambda_{\mathbf{z}'}}$ -expansion of $E_r(\mathfrak{s}; \mathbf{z})$ is $E_{r-1}(\mathfrak{s}; \mathbf{z}')$.

2.3. Drinfeld modular forms. Throughout this paper, the term “Drinfeld modular form of rank r ” refers to a Drinfeld modular form of type 0 for $\text{GL}_r(A)$. For any integer $r \geq 2$ and integer $w \geq 0$, we denote by $\mathcal{M}_w^{(r)}$ the $\mathbb{C}_\infty$ -vector space of Drinfeld modular forms of rank r and weight w , and by $\mathcal{M}^{(r)}$ the $\mathbb{C}_\infty$ -algebra of Drinfeld modular forms of rank r . Then

$$\mathcal{M}^{(r)} = \bigoplus_{w \geq 0} \mathcal{M}_w^{(r)}$$

forms a graded $\mathbb{C}_\infty$ -algebra (for the definition of Drinfeld modular forms and further details, see [BBP24]).

Now let $r \geq 1$ and let $\mathbb{C}_\infty[\tau]$ denote the twisted polynomial ring determined by the relation $\tau c = c^q \tau$ for all $c \in \mathbb{C}_\infty$. By the Drinfeld uniformization theorem (see [Gos96, §4.3]), for each $\mathbf{z} \in \Omega^r$, the A -lattice $\Lambda_{\mathbf{z}}$ determines a rank r Drinfeld A -module

$$\phi^{\Lambda_{\mathbf{z}}} : A \longrightarrow \mathbb{C}_\infty[\tau], \quad a \longmapsto \phi_a^{\Lambda_{\mathbf{z}}},$$

characterized by

$$\exp_{\Lambda_{\mathbf{z}}}(aX) = \phi_a^{\Lambda_{\mathbf{z}}}(\exp_{\Lambda_{\mathbf{z}}}(X))$$

for all $a \in A$. In particular,

$$(2.3.1) \quad \exp_{\Lambda_{\mathbf{z}}}(\theta X) = \phi_\theta^{\Lambda_{\mathbf{z}}}(\exp_{\Lambda_{\mathbf{z}}}(X)).$$

Now, write

$$\phi_\theta^{\Lambda_{\mathbf{z}}} = \theta + g_1(\mathbf{z})\tau + \dots + g_{r-1}(\mathbf{z})\tau^{r-1} + \Delta_r(\mathbf{z})\tau^r.$$

Notice that $\Delta_r(\mathbf{z})$ is nowhere vanishing.

In analogy with the classical setting, where the coefficients of the Weierstrass equation of an elliptic curve associated with a lattice generate the ring of modular forms for $\text{SL}_2(\mathbb{Z})$, we have the following results due to [BBP24].

Proposition 2.7. *For any integer $r \geq 2$, the following assertions hold.*

- (i) ([BBP24, Proposition 15.12 and Proposition 16.3]) *For $i = 1, \dots, r-1$, g_i is a Drinfeld modular form of weight $q^i - 1$, and Δ_r is a Drinfeld cusp form of weight $q^r - 1$.*
- (ii) ([BBP24, Theorem 17.5(a)]) *We have*

$$\mathcal{M}^{(r)} = \mathbb{C}_\infty[g_1, \dots, g_{r-1}, \Delta_r],$$

and $g_1, \dots, g_{r-1}, \Delta_r$ are algebraically independent over $\mathbb{C}_\infty$.

- (iii) ([BBP24, Theorem 17.5(a)]) *We have*

$$\mathcal{M}^{(r)} = \mathbb{C}_\infty[E_{q-1}^{(r)}, \dots, E_{q^r-1}^{(r)}],$$

and $E_{q-1}^{(r)}, \dots, E_{q^r-1}^{(r)}$ are algebraically independent over $\mathbb{C}_\infty$, where, for $s \geq 1$ with $q-1 \mid s$,

$$(2.3.2) \quad E_s^{(r)}(\mathbf{z}) = \sum_{0 \neq \lambda \in \Lambda_{\mathbf{z}}} \frac{1}{\lambda^s}$$

denotes the single Eisenstein series of rank r and weight s .

- (iv) ([BBP24, Proposition 15.3]) *For every $n \geq 0$, $e_n(\mathbf{z})$ is a Drinfeld modular form of weight $q^n - 1$.*
- (v) ([BBP24, (15.1) and (15.6)]) *We have*

$$K[g_1, \dots, g_{r-1}, \Delta_r] = K[e_i \mid i \geq 1] = K[E_{q-1}^{(r)}, \dots, E_{q^r-1}^{(r)}].$$

For any integer $r \geq 2$ and any subfield L of $\mathbb{C}_\infty$ containing K , we put

$$\mathcal{M}_L^{(r)} := L[g_1, \dots, g_{r-1}, \Delta_r] \subseteq \mathcal{M}^{(r)},$$

and denote by $\mathcal{M}_{L,w}^{(r)}$ its weight w homogeneous component. The following lemma shows that every element of $\mathcal{M}_{L,w}^{(r)}$ can be expressed as an L -linear combination of MES of weight w .

Lemma 2.8. *For any integer $r \geq 2$ and any subfield L of $\mathbb{C}_\infty$ containing K , we have*

$$\mathcal{M}_L^{(r)} = L[g_1, \dots, g_{r-1}, \Delta_r] \subseteq \mathcal{Z}_L^{(r)}.$$

Moreover, if $F \in L[g_1, \dots, g_{r-1}, \Delta_r]$ is a Drinfeld modular form of weight w , then

$$F \in \mathcal{Z}_{L,w}^{(r)}.$$

Proof. By (2.3.2), for every $s \geq 1$ with $q-1 \mid s$, one checks that

$$E_s^{(r)}(\mathbf{z}) = -E_r(s; \mathbf{z}),$$

and hence

$$E_s^{(r)} \in \mathcal{Z}_{L,s}^{(r)}.$$

By Proposition 2.7, we have an equality of graded L -algebras

$$L[g_1, \dots, g_{r-1}, \Delta_r] = L[E_{q-1}^{(r)}, \dots, E_{q^r-1}^{(r)}].$$

Thus, if $F \in \mathcal{M}_{L,w}^{(r)}$, then F is an L -linear combination of monomials in $E_{q-1}^{(r)}, \dots, E_{q^r-1}^{(r)}$ of total weight w . Since each $E_{q^i-1}^{(r)}$ belongs to $\mathcal{Z}_{L,q^i-1}^{(r)}$, it follows from (1.2.1) that

$$F \in \mathcal{Z}_{L,w}^{(r)}.$$

The first assertion follows by summing over all weights. $\square$

Remark 2.9. We fix a $(q-1)$ -st root of $(-\theta)$ and let

$$\tilde{\pi} = (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{\theta}{\theta^{q^i}}\right)^{-1} \in K_{\infty}((-\theta)^{\frac{1}{q-1}})$$

be a fixed fundamental period of the Carlitz A -module, namely, $\phi_{\theta}^{\Lambda_{\tilde{\pi}A}} = \theta + \tau$ (see [Gos96, §3.2]). When $r = 1$, we have the following.

- (i) The “rank one discriminant form” Δ_1 is given by $\Delta_1 = \tilde{\pi}^{q-1}$ since the Drinfeld A -module ϕ^A associated with the point $1 \in \Omega^1$ is determined by

$$\phi_{\theta}^A = \theta + \tilde{\pi}^{q-1}\tau.$$

By the Euler–Carlitz formula [Car35] (see also [Tha04, Theorem 5.2.1]), Δ_1 is a K -multiple of $\zeta_A(q-1)$. Hence

$$\Delta_1^m \in \mathcal{Z}_{K,m(q-1)}^{(1)} \quad (m \geq 0), \quad \text{and} \quad L[\Delta_1] \subseteq \mathcal{Z}_L^{(1)}$$

for any subfield L of $\mathbb{C}_{\infty}$ containing K .

- (ii) For every $n \geq 0$, the Carlitz exponential formula (see [Gos96, §3.2]) gives

$$e_n(1) = \frac{\tilde{\pi}^{q^n-1}}{D_n} = \frac{1}{D_n} \Delta_1^{(q^n-1)/(q-1)} \in \mathcal{Z}_{K,q^n-1}^{(1)},$$

where $D_0 = 1$ and $D_n = \prod_{j=0}^{n-1} (\theta^{q^n} - \theta^{q^j})$.

3. PROOF OF THEOREM 1.5

In this section, we prove Theorem 1.5. In §3.1, we first compute the coefficients in the $t_{\Lambda_{z'}}$ -expansion of MES. Our main goal is to determine the precise “weight” carried by each coefficient in the $t_{\Lambda_{z'}}$ -expansion of MES (see Theorem 3.6). This enables us to prove Theorem 1.5 (i) and (ii) in §3.2 and §3.3, respectively.

3.1. t -expansion coefficients of multiple Eisenstein series. Fix $r \geq 1$ and a subfield L of $\mathbb{C}_{\infty}$ containing K . Throughout this section, we always write

$$\mathbf{z} = (z_1, \mathbf{z}') \in \Omega^{r+1}, \quad \mathbf{z}' = (z_2, \dots, z_r, 1) \in \Omega^r.$$

We study the coefficients occurring in the $t_{\Lambda_{z'}}$ -expansion of MES of rank $r+1$. Note that the natural projection

$$\Omega^{r+1} \longrightarrow \Omega^r, \quad (z_1, \mathbf{z}') \longmapsto \mathbf{z}',$$

induces an injective $\mathbb{C}_{\infty}$ -algebra homomorphism

$$\mathcal{O}(\Omega^r) \hookrightarrow \mathcal{O}(\Omega^{r+1}),$$

and we identify $\mathcal{O}(\Omega^r)$ with its image under this map.

We first consider the localization

$$\mathcal{Z}_L^{(r)}[\Delta_r^{-1}] = \sum_{w \in \mathbb{Z}} \mathcal{L}_{L,w}^{(r)},$$

where

$$\mathcal{L}_{L,w}^{(r)} := \left\{ \frac{F}{\Delta_r^M} \mid M \geq 0 \text{ and } F \in \mathcal{Z}_{L,w+M(q^r-1)}^{(r)} \text{ with } w + M(q^r-1) \geq 0 \right\}.$$

Notice that by (1.2.1), for any integers w_1, w_2 ,

$$\mathcal{L}_{L,w_1}^{(r)} \cdot \mathcal{L}_{L,w_2}^{(r)} \subseteq \mathcal{L}_{L,w_1+w_2}^{(r)}.$$

Lemma 3.1. Fix an integer $r \geq 1$ and denote by $\mathbf{w}$ an arbitrary point varying in Ω^r . For $f \in A_+$ with $\deg f = d$, write

$$\phi_f^{\Lambda_{\mathbf{w}}}(X) = \sum_{i=0}^{rd} g_{f,i}(\mathbf{w}) X^{q^i} \in \mathcal{O}(\Omega^r)[X].$$

Then

$$g_{f,i} \in K[g_1, \dots, g_{r-1}, \Delta_r]$$

is of weight $q^i - 1$ for every $0 \leq i \leq rd$. Moreover,

$$g_{f,rd} = \Delta_r^{(q^{rd}-1)/(q^r-1)}.$$

Proof. For $r = 1$, the assertions follow from Remark 2.9 and the fact that $\phi^A : A \rightarrow \mathbb{C}_\infty[\tau]$ is an $\mathbb{F}_q$ -algebra homomorphism. For $r \geq 2$, see [BBP24, Propositions 15.12 and 16.4]. $\square$

Lemma 3.2. Fix an integer $r \geq 1$ and denote by $\mathbf{z} = (z_1, \mathbf{z}')$ an arbitrary point varying in Ω^{r+1} . For $f \in A_+$, write

$$t_{\Lambda_{\mathbf{z}}}(fz_1) = \sum_{N \geq 0} \mathcal{A}_{f,N}(\mathbf{z}') \cdot t_{\Lambda_{\mathbf{z}}}(z_1)^N.$$

Then

$$\mathcal{A}_{f,N}(\mathbf{z}') \in \mathcal{L}_{K,1-N}^{(r)}$$

for all $N \geq 0$.

Proof. Let $d = \deg f$. We note that from (2.1.1) and (2.3.1), we have

$$\begin{aligned} t_{\Lambda_{\mathbf{z}}}(fz_1) &= \frac{1}{\exp_{\Lambda_{\mathbf{z}}}(fz_1)} = \frac{1}{\phi_f^{\Lambda_{\mathbf{z}}}(t_{\Lambda_{\mathbf{z}}}(z_1)^{-1})} \\ &= \frac{1}{\sum_{i=0}^{rd} g_{f,i}(\mathbf{z}') t_{\Lambda_{\mathbf{z}}}(z_1)^{-q^i}} \\ &= \frac{t_{\Lambda_{\mathbf{z}}}(z_1)^{q^{rd}}}{g_{f,rd}(\mathbf{z}')} \left(1 + \sum_{i=0}^{rd-1} \frac{g_{f,i}(\mathbf{z}')}{g_{f,rd}(\mathbf{z}')} t_{\Lambda_{\mathbf{z}}}(z_1)^{q^{rd}-q^i} \right)^{-1}. \end{aligned}$$

By Lemma 3.1, together with Remark 2.9 for $r = 1$ and Lemma 2.8 for $r \geq 2$, we have

$$g_{f,i}(\mathbf{z}') \in \mathcal{Z}_{K,q^i-1}^{(r)}, \quad g_{f,rd}(\mathbf{z}')^{-1} \in \mathcal{L}_{K,1-q^{rd}}^{(r)},$$

and hence

$$\frac{g_{f,i}(\mathbf{z}')}{g_{f,rd}(\mathbf{z}')} \in \mathcal{L}_{K,q^i-q^{rd}}^{(r)} \quad (0 \leq i \leq rd-1).$$

Expanding the inverse as a formal power series gives

$$\left(1 + \sum_{i=0}^{rd-1} \frac{g_{f,i}(\mathbf{z}')}{g_{f,rd}(\mathbf{z}')} t_{\Lambda_{\mathbf{z}'}}(z_1)^{q^{rd}-q^i}\right)^{-1} = \sum_{m \geq 0} (-1)^m \left(\sum_{i=0}^{rd-1} \frac{g_{f,i}(\mathbf{z}')}{g_{f,rd}(\mathbf{z}')} t_{\Lambda_{\mathbf{z}'}}(z_1)^{q^{rd}-q^i} \right)^m.$$

Consider a monomial occurring on the right-hand side,

$$\prod_{j=1}^m \frac{g_{f,i_j}(\mathbf{z}')}{g_{f,rd}(\mathbf{z}')} t_{\Lambda_{\mathbf{z}'}}(z_1)^{q^{rd}-q^{i_j}}.$$

Its coefficient belongs to $\mathcal{L}_{K, \sum_{j=1}^m (q^{i_j}-q^{rd})}^{(r)}$, while its power of $t_{\Lambda_{\mathbf{z}'}}(z_1)$ is

$$\sum_{j=1}^m (q^{rd} - q^{i_j}).$$

Thus, if such a monomial contributes to the coefficient of t^N in $t_{\Lambda_{\mathbf{z}'}}(fz_1)$, then

$$N = q^{rd} + \sum_{j=1}^m (q^{rd} - q^{i_j}),$$

and its coefficient belongs to

$$\mathcal{L}_{K, 1-q^{rd} + \sum_{j=1}^m (q^{i_j}-q^{rd})}^{(r)} = \mathcal{L}_{K, 1-N}^{(r)}.$$

Therefore every term contributing to $\mathcal{A}_{f,N}(\mathbf{z}')$ lies in $\mathcal{L}_{K, 1-N}^{(r)}$, and hence

$$\mathcal{A}_{f,N}(\mathbf{z}') \in \mathcal{L}_{K, 1-N}^{(r)}$$

for all $N \geq 0$. □

Lemma 3.3. Fix an integer $r \geq 1$ and denote by $\mathbf{w}$ an arbitrary point varying in Ω^r . For $s \geq 1$, write

$$G_s^{\Lambda_{\mathbf{w}}}(X) = \sum_{i=1}^s \mathcal{B}_{s,i}(\mathbf{w}) X^i \in \mathcal{O}(\Omega^r)[X].$$

Then

$$\mathcal{B}_{s,i} \in \mathcal{L}_{K,s-i}^{(r)}$$

for every $1 \leq i \leq s$.

Proof. Recall that the coefficient $e_n(\mathbf{w}) \in \mathcal{O}(\Omega^r)$ of $\exp_{\Lambda_{\mathbf{w}}}(X)$ is defined in (2.2.1). By Remark 2.9 for $r = 1$ and Proposition 2.7 and Lemma 2.8 for $r \geq 2$, we have

$$e_n(\mathbf{w}) \in \mathcal{Z}_{K,q^n-1}^{(r)}.$$

According to [Gek88, (3.8)], the Goss polynomial associated with $\Lambda_{\mathbf{w}}$ is given by

$$G_s^{\Lambda_{\mathbf{w}}}(X) = \sum_{j=0}^{s-1} \sum_{\underline{i}} \binom{j}{\underline{i}} e^{\underline{i}}(\mathbf{w}) X^{j+1},$$

where the inner sum runs over all tuples $\underline{i} = (i_0, \dots, i_{s-1})$ of non-negative integers satisfying

$$\sum_{l=0}^{s-1} i_l = j \quad \text{and} \quad \sum_{l=0}^{s-1} i_l q^l = s-1,$$

and

$$e^{\underline{i}}(\mathbf{w}) := e_0(\mathbf{w})^{i_0} \cdots e_{s-1}(\mathbf{w})^{i_{s-1}}.$$

Hence, by (1.2.1),

$$e^{\underline{i}}(\mathbf{w}) \in \mathcal{Z}_{K, \sum_{l=0}^{s-1} i_l (q^l - 1)}^{(r)} = \mathcal{Z}_{K, s-1-j}^{(r)}.$$

Taking $j = i - 1$, we obtain

$$\mathcal{B}_{s,i}(\mathbf{w}) \in \mathcal{Z}_{K, s-i}^{(r)} \subseteq \mathcal{L}_{K, s-i}^{(r)}.$$

□

Lemma 3.4. Fix an integer $r \geq 1$ and denote by $\mathbf{z} = (z_1, \mathbf{z}')$ an arbitrary point varying in Ω^{r+1} . For $f \in A_+$ and $s \geq 1$, write

$$G_s^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(fz_1)) = \sum_{N \geq 0} \mathcal{C}_{s,f,N}(\mathbf{z}') \cdot t_{\Lambda_{\mathbf{z}'}}(z_1)^N.$$

Then

$$\mathcal{C}_{s,f,N}(\mathbf{z}') \in \mathcal{L}_{K, s-N}^{(r)}$$

for all $N \geq 0$.

Proof. Using the same notation as in Lemma 3.2 and Lemma 3.3, we see that

$$G_s^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(fz_1)) = \sum_{i=1}^s \mathcal{B}_{s,i}(\mathbf{z}') \left(\sum_{M \geq 0} \mathcal{A}_{f,M}(\mathbf{z}') \cdot t_{\Lambda_{\mathbf{z}'}}(z_1)^M \right)^i$$

Comparing the coefficient of $t_{\Lambda_{\mathbf{z}'}}(z_1)^N$, we obtain

$$\begin{aligned} \mathcal{C}_{s,f,N}(\mathbf{z}') &= \sum_{i=1}^s \mathcal{B}_{s,i}(\mathbf{z}') \left(\sum_{M_1 + \dots + M_i = N} \prod_{j=1}^i \mathcal{A}_{f,M_j}(\mathbf{z}') \right) \\ &= \sum_{i=1}^s \sum_{M_1 + \dots + M_i = N} \mathcal{B}_{s,i}(\mathbf{z}') \prod_{j=1}^i \mathcal{A}_{f,M_j}(\mathbf{z}'). \end{aligned}$$

By Lemma 3.2 and Lemma 3.3, for each $1 \leq i \leq s$, we have

$$\mathcal{B}_{s,i}(\mathbf{z}') \prod_{j=1}^i \mathcal{A}_{f,M_j}(\mathbf{z}') \in \mathcal{L}_{K, s-i}^{(r)} \cdot \mathcal{L}_{K, 1-M_1}^{(r)} \cdots \mathcal{L}_{K, 1-M_i}^{(r)} \subseteq \mathcal{L}_{K, s-N}^{(r)}.$$

Therefore, $\mathcal{C}_{s,f,N}(\mathbf{z}') \in \mathcal{L}_{K, s-N}^{(r)}$, as desired.

□

Proposition 3.5. Fix an integer $r \geq 1$ and denote by $\mathbf{z} = (z_1, \mathbf{z}')$ an arbitrary point varying in Ω^{r+1} . For any index $\mathfrak{s}$, write

$$G_{r+1}(\mathfrak{s}; \mathbf{z}) = \sum_{N \geq 0} \alpha_{\mathfrak{s}, N}(\mathbf{z}') \cdot t_{\Lambda_{\mathbf{z}'}}(z_1)^N.$$

Then

$$\alpha_{\mathfrak{s}, N}(\mathbf{z}') \in \mathcal{L}_{K, \text{wt}(\mathfrak{s}) - N}^{(r)}$$

for all $N \geq 0$.

Proof. We may assume that $\mathfrak{s} = (s_1, \dots, s_m)$ is nonempty. Fix $f_1, \dots, f_m \in A_+$ with $\deg f_1 > \dots > \deg f_m$. Then by Lemma 3.4, we have

$$\begin{aligned} & G_{s_1}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(f_1 z_1)) \cdots G_{s_m}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(f_m z_1)) \\ &= \left(\sum_{N_1=0}^{\infty} \mathcal{C}_{s_1, f_1, N_1}(\mathbf{z}') t_{\Lambda_{\mathbf{z}'}}(z_1)^{N_1} \right) \cdots \left(\sum_{N_m=0}^{\infty} \mathcal{C}_{s_m, f_m, N_m}(\mathbf{z}') t_{\Lambda_{\mathbf{z}'}}(z_1)^{N_m} \right) \\ &= \sum_{N=0}^{\infty} \left(\sum_{\substack{N_1 + \dots + N_m = N \\ N_1, \dots, N_m \geq 0}} \mathcal{C}_{s_1, f_1, N_1}(\mathbf{z}') \cdots \mathcal{C}_{s_m, f_m, N_m}(\mathbf{z}') \right) t_{\Lambda_{\mathbf{z}'}}(z_1)^N \end{aligned}$$

and

$$\mathcal{C}_{s_1, f_1, N_1}(\mathbf{z}') \cdots \mathcal{C}_{s_m, f_m, N_m}(\mathbf{z}') \in \mathcal{L}_{K, s_1 - N_1}^{(r)} \cdots \mathcal{L}_{K, s_m - N_m}^{(r)} \subseteq \mathcal{L}_{K, \text{wt}(\mathfrak{s}) - N}^{(r)}.$$

Now, note that by (2.2.2), for all $s \geq 1$ and $f \in A_+$, we have

$$\text{ord}_{r+1}(G_s(t_{\Lambda_{\mathbf{z}'}}(f z_1))) \geq q^r \deg f.$$

In particular, if $\deg f > \frac{1}{r} \log_q(N+1)$, then $\text{ord}_{r+1}(G_s(t_{\Lambda_{\mathbf{z}'}}(f z_1))) > N$. Hence, for all $N \geq 0$,

$$\begin{aligned} & G_{r+1}(\mathfrak{s}; \mathbf{z}) \\ &\equiv \sum_{\substack{f_1, \dots, f_m \in A_+ \\ \frac{1}{r} \log_q(N+1) \geq \deg f_1 > \dots > \deg f_m}} G_{s_1}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(f_1 z_1)) \cdots G_{s_m}^{\Lambda_{\mathbf{z}'}}(t_{\Lambda_{\mathbf{z}'}}(f_m z_1)) \pmod{t_{\Lambda_{\mathbf{z}'}}(z_1)^{N+1}}. \end{aligned}$$

Therefore, we obtain

$$\alpha_{\mathfrak{s}, N}(\mathbf{z}') = \sum_{\substack{f_1, \dots, f_m \in A_+ \\ \frac{1}{r} \log_q(N+1) \geq \deg f_1 > \dots > \deg f_m}} \sum_{\substack{N_1 + \dots + N_m = N \\ N_1, \dots, N_m \geq 0}} \mathcal{C}_{s_1, f_1, N_1}(\mathbf{z}') \cdots \mathcal{C}_{s_m, f_m, N_m}(\mathbf{z}') \in \mathcal{L}_{K, \text{wt}(\mathfrak{s}) - N}^{(r)}.$$

□

Theorem 3.6. Fix an integer $r \geq 1$ and denote by $\mathbf{z} = (z_1, \mathbf{z}')$ an arbitrary point varying in Ω^{r+1} . For any index $\mathfrak{s}$, write

$$E_{r+1}(\mathfrak{s}; \mathbf{z}) = \sum_{N \geq 0} \beta_{\mathfrak{s}, N}(\mathbf{z}') \cdot t_{\Lambda_{\mathbf{z}'}}(z_1)^N.$$

Then

$$\beta_{\mathfrak{s}, N}(\mathbf{z}') \in \mathcal{L}_{K, \text{wt}(\mathfrak{s}) - N}^{(r)}$$

for all $N \geq 0$.

Proof. We may assume that $\mathfrak{s} = (s_1, \dots, s_m)$ is nonempty. By Proposition 2.6, we have

$$E_{r+1}(\mathfrak{s}; \mathbf{z}) = \sum_{i=0}^m E_r(\mathfrak{s}^{(i)}; \mathbf{z}') G_{r+1}(\mathfrak{s}_{(i)}; \mathbf{z}).$$

For each $0 \leq i \leq m$, write

$$G_{r+1}(\mathfrak{s}_{(i)}; \mathbf{z}) = \sum_{N \geq 0} \alpha_{\mathfrak{s}_{(i)}, N}(\mathbf{z}') t_{\Lambda_{\mathbf{z}'}}(z_1)^N.$$

By Proposition 3.5, we have

$$\alpha_{\mathfrak{s}_{(i)}, N}(\mathbf{z}') \in \mathcal{L}_{K, \text{wt}(\mathfrak{s}_{(i)}) - N}^{(r)}.$$

Moreover,

$$E_r(\mathfrak{s}^{(i)}; \mathbf{z}') \in \mathcal{Z}_{K, \text{wt}(\mathfrak{s}^{(i)})}^{(r)} \subseteq \mathcal{L}_{K, \text{wt}(\mathfrak{s}^{(i)})}^{(r)}.$$

Hence

$$E_r(\mathfrak{s}^{(i)}; \mathbf{z}') \alpha_{\mathfrak{s}_{(i)}, N}(\mathbf{z}') \in \mathcal{L}_{K, \text{wt}(\mathfrak{s}) - N}^{(r)}.$$

Comparing the coefficients of $t_{\Lambda_{\mathbf{z}'}}(z_1)^N$, we obtain

$$\beta_{\mathfrak{s}, N}(\mathbf{z}') = \sum_{i=0}^m E_r(\mathfrak{s}^{(i)}; \mathbf{z}') \alpha_{\mathfrak{s}_{(i)}, N}(\mathbf{z}') \in \mathcal{L}_{K, \text{wt}(\mathfrak{s}) - N}^{(r)},$$

as desired. $\square$

3.2. Proof of Theorem 1.5 (i). We proceed by induction on r . The case $r = 1$ is exactly Theorem 1.3 (i). Let $r \geq 1$ and assume that the assertions hold for r . Suppose that we have a $\overline{K}$ -linear relation

$$(3.2.1) \quad \sum_{i=1}^m \sum_{\text{wt}(\mathfrak{s})=w_i} c_{\mathfrak{s}} E_{r+1}(\mathfrak{s}; \mathbf{z}) = 0$$

for some distinct $w_1, \dots, w_m \geq 0$ and $c_{\mathfrak{s}} \in \overline{K}$. It suffices to show that, for each $1 \leq i \leq m$,

$$\sum_{\text{wt}(\mathfrak{s})=w_i} c_{\mathfrak{s}} E_{r+1}(\mathfrak{s}; \mathbf{z}) = 0.$$

By (3.2.1), we obtain

$$\sum_{N \geq 0} \sum_{i=1}^m \sum_{\text{wt}(\mathfrak{s})=w_i} c_{\mathfrak{s}} \beta_{\mathfrak{s}, N}(\mathbf{z}') \cdot t_{\Lambda_{\mathbf{z}'}}(z_1)^N = 0.$$

By the uniqueness of the $t_{\Lambda_{\mathbf{z}'}}(z_1)$ -expansion (see [BBP24, Proposition 5.4]), it follows that

$$(3.2.2) \quad \sum_{i=1}^m \sum_{\text{wt}(\mathfrak{s})=w_i} c_{\mathfrak{s}} \beta_{\mathfrak{s}, N}(\mathbf{z}') = 0$$

for all $N \geq 0$.

Fix $N \geq 0$. By Theorem 3.6,

$$\beta_{\mathfrak{s}, N}(\mathbf{z}') \in \mathcal{L}_{K, \text{wt}(\mathfrak{s}) - N}^{(r)}.$$

Hence, after choosing $M \gg 1$ sufficiently large, we may write

$$\beta_{\mathfrak{s},N}(\mathbf{z}') = \Delta_r^{-M} \Xi_{\mathfrak{s},N}(\mathbf{z}')$$

for all $\mathfrak{s}$ occurring in (3.2.2), where

$$\Xi_{\mathfrak{s},N}(\mathbf{z}') \in \mathcal{Z}_{\bar{K}, \text{wt}(\mathfrak{s})-N+M(q^r-1)}^{(r)}.$$

Thus, after multiplying (3.2.2) by Δ_r^M , we obtain

$$\sum_{i=1}^m \sum_{\text{wt}(\mathfrak{s})=w_i} c_{\mathfrak{s}} \Xi_{\mathfrak{s},N}(\mathbf{z}') = 0.$$

For each i , note that the inner sum belongs to $\mathcal{Z}_{\bar{K}, w_i-N+M(q^r-1)}^{(r)}$ and the weights

$$w_i - N + M(q^r - 1)$$

are all distinct. Therefore, by the induction hypothesis,

$$\sum_{\text{wt}(\mathfrak{s})=w_i} c_{\mathfrak{s}} \Xi_{\mathfrak{s},N}(\mathbf{z}') = 0$$

for each $1 \leq i \leq m$. Dividing by Δ_r^M , we obtain

$$\sum_{\text{wt}(\mathfrak{s})=w_i} c_{\mathfrak{s}} \beta_{\mathfrak{s},N}(\mathbf{z}') = 0$$

for every $N \geq 0$ and every $1 \leq i \leq m$. Hence, by the $t_{\Lambda_{\mathbf{z}'}}(z_1)$ -expansion,

$$\sum_{\text{wt}(\mathfrak{s})=w_i} c_{\mathfrak{s}} E_{r+1}(\mathfrak{s}; \mathbf{z}) = 0$$

for each $1 \leq i \leq m$, as desired.

3.3. Proof of Theorem 1.5 (ii). The surjectivity of the natural map

$$\mathcal{Z}_K^{(r)} \otimes_K \bar{K} \longrightarrow \mathcal{Z}_{\bar{K}}^{(r)}$$

is immediate from the definition. We prove its injectivity by induction on r . The case $r = 1$ follows from Theorem 1.3 (ii). Let $r \geq 1$ and assume that the assertion holds in rank r . It suffices to show that every K -linearly independent family

$$F_1, \dots, F_m \in \mathcal{Z}_K^{(r+1)}$$

remains linearly independent over $\bar{K}$.

By Theorem 3.6, for each $1 \leq i \leq m$, we may write

$$F_i(\mathbf{z}) = \sum_{N \geq 0} \Delta_r^{-M(N)} \gamma_{i,N}(\mathbf{z}') t_{\Lambda_{\mathbf{z}'}}(z_1)^N,$$

where, for each $N \geq 0$, the integer $M(N) \geq 0$ is chosen sufficiently large so that

$$\gamma_{i,N}(\mathbf{z}') \in \mathcal{Z}_K^{(r)}$$

for all $1 \leq i \leq m$. For each $N \geq 0$, define the K -linear map

$$\psi_N : K^{\oplus m} \longrightarrow \mathcal{Z}_K^{(r)}, \quad (c_1, \dots, c_m) \longmapsto \sum_{i=1}^m c_i \gamma_{i,N}.$$

Since Δ_r is nowhere vanishing, the uniqueness of the $t_{\Lambda_{z'}}$ -expansion and the K -linear independence of $F_1, \dots, F_m$ imply that

$$\bigcap_{N \geq 0} \ker \psi_N = \{0\}.$$

Since $K^{\oplus m}$ is finite-dimensional, there exists $N_0 \geq 0$ such that

$$\bigcap_{N=0}^{N_0} \ker \psi_N = \{0\}.$$

Hence the K -linear map

$$\Psi : K^{\oplus m} \longrightarrow \left(\mathcal{Z}_K^{(r)} \right)^{\oplus (N_0+1)}, \quad \mathbf{c} \longmapsto \left(\psi_0(\mathbf{c}), \dots, \psi_{N_0}(\mathbf{c}) \right)$$

is injective.

By the induction hypothesis,

$$\mathcal{Z}_K^{(r)} \otimes_K \bar{K} \cong \mathcal{Z}_{\bar{K}}^{(r)}.$$

Since $\bar{K}$ is flat over K , extending scalars from K to $\bar{K}$ shows that

$$\Psi_{\bar{K}} : \bar{K}^{\oplus m} \longrightarrow \left(\mathcal{Z}_{\bar{K}}^{(r)} \right)^{\oplus (N_0+1)},$$

defined by

$$(c_1, \dots, c_m) \longmapsto \left(\sum_{i=1}^m c_i \gamma_{i,0}, \dots, \sum_{i=1}^m c_i \gamma_{i,N_0} \right),$$

is injective.

Now suppose that $c_1, \dots, c_m \in \bar{K}$ satisfy

$$\sum_{i=1}^m c_i F_i = 0.$$

Comparing the coefficients of $t_{\Lambda_{z'}}(z_1)^N$ and multiplying by $\Delta_r^{M(N)}$, we obtain

$$\sum_{i=1}^m c_i \gamma_{i,N} = 0$$

for every $N \geq 0$. In particular, we have

$$(c_1, \dots, c_m) \in \ker \Psi_{\bar{K}} = 0.$$

Thus, $F_1, \dots, F_m$ are $\bar{K}$ -linearly independent. This proves the injectivity of

$$\mathcal{Z}_K^{(r+1)} \otimes_K \bar{K} \longrightarrow \mathcal{Z}_{\bar{K}}^{(r+1)},$$

and completes the induction.

APPENDIX A. THEOREM 1.5 OVER $\mathbb{C}_\infty$

In this appendix, we aim to prove the following stronger version of Theorem 1.5 by adopting the key ideas from [BK26], particularly the notion of “shifted” MES in Definition A.5.

Theorem A.1. *Fix a positive integer $r \geq 2$. Then the following results hold.*

(i) $\mathcal{Z}_{\mathbb{C}_\infty}^{(r)}$ is a graded algebra, i.e.,

$$\mathcal{Z}_{\mathbb{C}_\infty}^{(r)} = \bigoplus_{w=0}^{\infty} \mathcal{Z}_{\mathbb{C}_\infty, w}^{(r)}.$$

(ii) The natural map $\mathcal{Z}_K^{(r)} \otimes_K \mathbb{C}_\infty \rightarrow \mathcal{Z}_{\mathbb{C}_\infty}^{(r)}$ is an isomorphism of $\mathbb{C}_\infty$ -algebras.

Note that when $r = 1$, MES are Thakur’s MZVs, and the corresponding base-change statement over $\mathbb{C}_\infty$ does not hold. We first mention the following immediate consequence concerning the non-vanishing of rank two MGS.

Corollary A.2. *For any index $\mathfrak{s}$,*

$$G_2(\mathfrak{s}; \mathbf{z}) \neq 0.$$

Proof. If $G_2(\mathfrak{s}; \mathbf{z}) = 0$, then

$$E_2(\mathfrak{s}; \mathbf{z}) = \sum_{i=0}^{\text{dep}(\mathfrak{s})-1} \zeta_A(\mathfrak{s}^{(i)}) G_2(\mathfrak{s}_{(i)}; \mathbf{z}) \in \sum_{w=0}^{\text{wt}(\mathfrak{s})-1} \mathcal{Z}_{\mathbb{C}_\infty, w}^{(2)}.$$

By Theorem A.1, this implies

$$E_2(\mathfrak{s}; \mathbf{z}) = 0.$$

On the other hand, by [CCHT26, Proposition 2.10], the constant term of the $t_{\Lambda_{\mathbf{z}'}}(z_1)$ -expansion of $E_2(\mathfrak{s}; \mathbf{z})$ is $E_1(\mathfrak{s}) = \zeta_A(\mathfrak{s})$, which is nonzero by [Tha09, Theorem 4]. This is a contradiction. $\square$

A.1. Proof of Theorem A.1 (i). For $N \geq 1$, we let

$$\gamma_N = \left(\begin{array}{c|cc} I_{r-2} & 0 & 0 \\ \hline 0 & \theta^N & -1 \\ 0 & 1 & 0 \end{array} \right) \in \text{SL}_r(A) \subseteq \text{GL}_r(A),$$

and define a new partial order $\succ_N$ on A^r as follows.

Definition A.3. For $f, g \in A^r$, we define

$$f \succ_N 0 \iff f \gamma_N \succ 0,$$

and

$$f \succ_N g \iff f \gamma_N \succ g \gamma_N.$$

For $\mathbf{z} = (z_1, \dots, z_r) \in \mathbb{C}_\infty^r$, we put

$$|\mathbf{z}| := \max_{1 \leq i \leq r} |z_i|.$$

Lemma A.4. Let $\mathbf{f}, \mathbf{g} \in A^r$ be nonzero with $|\mathbf{f}|, |\mathbf{g}| < q^N$. Then

- (i) $\mathbf{f} \succ_N 0$ if and only if $\mathbf{f} \succ 0$.
- (ii) $\mathbf{f} \succ_N \mathbf{g}$ if and only if $\mathbf{f} \succ \mathbf{g}$.

Proof. Write $\mathbf{f} = (f_1, \dots, f_r)$. We have

$$\mathbf{f}\gamma_N = (f_1, \dots, f_{r-2}, \theta^N f_{r-1} + f_r, -f_{r-1}).$$

If $\ell(\mathbf{f}) \leq r-2$, the first nonzero coordinate of $\mathbf{f}\gamma_N$ is the same as that of $\mathbf{f}$, so the first assertion is immediate. If $\ell(\mathbf{f}) = r-1$, then $\deg f_r < N$, while

$$\deg(\theta^N f_{r-1}) = N + \deg f_{r-1} \geq N.$$

Hence $\theta^N f_{r-1} + f_r$ is monic if and only if f_{r-1} is monic. Finally, if $\ell(\mathbf{f}) = r$, then

$$\mathbf{f}\gamma_N = (0, \dots, 0, f_r, 0),$$

so again $\mathbf{f} \succ_N 0$ if and only if $\mathbf{f} \succ 0$. This proves (i). The proof of (ii) follows from a similar argument. $\square$

Using the partial order $\succ_N$, we define a “shifted” MES as follows.

Definition A.5. For any nonempty index $\mathfrak{s} = (s_1, \dots, s_m)$, we define

$$E_r^{(N)}(\mathfrak{s}; \mathbf{z}) = \sum_{\substack{\mathbf{f}_1, \dots, \mathbf{f}_m \in A^r \\ \mathbf{f}_1 \succ_N \dots \succ_N \mathbf{f}_m \succ_N 0}} \frac{1}{\langle \mathbf{f}_1, \mathbf{z} \rangle^{s_1} \dots \langle \mathbf{f}_m, \mathbf{z} \rangle^{s_m}}.$$

By convention, we put $E_r^{(N)}(\emptyset; \mathbf{z}) = 1$.

The next proposition compares $E_r^{(N)}(\mathfrak{s}; \mathbf{z})$ with the original MES. It shows that the shifted series converges to $E_r(\mathfrak{s}; \mathbf{z})$ as $N \rightarrow \infty$ and relates it to the action of γ_N .

Proposition A.6. Let $\mathfrak{s}$ be an index.

- (i) For every $\mathbf{z} \in \Omega^r$, there exists a constant $c(\mathbf{z}) > 0$ such that

$$\left| E_r^{(N)}(\mathfrak{s}; \mathbf{z}) - E_r(\mathfrak{s}; \mathbf{z}) \right| \leq c(\mathbf{z})q^{-N}$$

for every $N \geq 1$.

- (ii) For $\mathbf{z} = (z_1, \dots, z_{r-1}, 1) \in \Omega^r$, put

$$j_N(\mathbf{z}) := \theta^N - z_{r-1}.$$

Then

$$\mathbf{z}_N := \left(\frac{z_1}{j_N(\mathbf{z})}, \dots, \frac{z_{r-2}}{j_N(\mathbf{z})}, \frac{1}{j_N(\mathbf{z})}, 1 \right) \in \Omega^r,$$

and

$$E_r(\mathfrak{s}; \mathbf{z}_N) = j_N(\mathbf{z})^{\text{wt}(\mathfrak{s})} E_r^{(N)}(\mathfrak{s}; \mathbf{z}).$$

Proof. We may assume $\mathfrak{s} = (s_1, \dots, s_m)$ is nonempty. By Lemma A.4, the two summation sets defining $E_r^{(N)}(\mathfrak{s}; \mathbf{z})$ and $E_r(\mathfrak{s}; \mathbf{z})$ agree whenever

$$|\mathbf{f}_1|, \dots, |\mathbf{f}_m| < q^N.$$

Hence their difference is supported on tuples for which $|\mathbf{f}_i| \geq q^N$ for some $1 \leq i \leq m$. On the other hand, notice that for each fixed $\mathbf{z} \in \Omega^r$, the K_∞ -linear map

$$K_\infty^r \longrightarrow \mathbb{C}_\infty, \quad \mathbf{f} \longmapsto \langle \mathbf{f}, \mathbf{z} \rangle,$$

is injective. Since K_∞^r is finite-dimensional, there exists $c_0(\mathbf{z}) > 0$ such that

$$|\langle \mathbf{f}, \mathbf{z} \rangle| \geq c_0(\mathbf{z})|\mathbf{f}| \quad (\mathbf{f} \in K_\infty^r).$$

Thus, a standard estimate shows that there exists a constant $c(\mathbf{z}) > 0$, independent of N , such that

$$\left| E_r^{(N)}(\mathfrak{s}; \mathbf{z}) - E_r(\mathfrak{s}; \mathbf{z}) \right| \leq c(\mathbf{z})q^{-N}.$$

This proves (i).

For (ii), one checks that

$$\gamma_N \mathbf{z}_N^{\text{tr}} = j_N(\mathbf{z})^{-1} \mathbf{z}^{\text{tr}},$$

which implies $\mathbf{z}_N \in \Omega^r$. It follows that

$$\begin{aligned} E_r(\mathfrak{s}; \mathbf{z}_N) &= \sum_{\substack{\mathbf{f}_1, \dots, \mathbf{f}_m \in A^r \\ \mathbf{f}_1 \succ \dots \succ \mathbf{f}_m \succ 0}} \frac{1}{(\mathbf{f}_1 \mathbf{z}_N^{\text{tr}})^{s_1} \dots (\mathbf{f}_m \mathbf{z}_N^{\text{tr}})^{s_m}} \\ &= j_N(\mathbf{z})^{\text{wt}(\mathfrak{s})} \sum_{\substack{\mathbf{f}_1, \dots, \mathbf{f}_m \in A^r \\ \mathbf{f}_1 \succ \dots \succ \mathbf{f}_m \succ 0}} \frac{1}{(\mathbf{f}_1 \gamma_N^{-1} \mathbf{z}^{\text{tr}})^{s_1} \dots (\mathbf{f}_m \gamma_N^{-1} \mathbf{z}^{\text{tr}})^{s_m}} \\ &= j_N(\mathbf{z})^{\text{wt}(\mathfrak{s})} \sum_{\substack{\mathbf{f}_1, \dots, \mathbf{f}_m \in A^r \\ \mathbf{f}_1 \succ \dots \succ \mathbf{f}_m \succ 0}} \frac{1}{\langle \mathbf{f}_1 \gamma_N^{-1}, \mathbf{z} \rangle^{s_1} \dots \langle \mathbf{f}_m \gamma_N^{-1}, \mathbf{z} \rangle^{s_m}} \\ &= j_N(\mathbf{z})^{\text{wt}(\mathfrak{s})} \sum_{\substack{\mathbf{f}_1, \dots, \mathbf{f}_m \in A^r \\ \mathbf{f}_1 \succ_N \dots \succ_N \mathbf{f}_m \succ_N 0}} \frac{1}{\langle \mathbf{f}_1, \mathbf{z} \rangle^{s_1} \dots \langle \mathbf{f}_m, \mathbf{z} \rangle^{s_m}} \\ &= j_N(\mathbf{z})^{\text{wt}(\mathfrak{s})} E_r^{(N)}(\mathfrak{s}; \mathbf{z}), \end{aligned}$$

where the fourth equality follows by the change of variables $\mathbf{f}_i \mapsto \mathbf{f}_i \gamma_N^{-1}$ and the definition of $\succ_N$. This proves (ii). $\square$

Proof of Theorem A.1 (i). Suppose that

$$F_0 + F_1 + \dots + F_w = 0,$$

where $F_i \in \mathcal{Z}_{\mathbb{C}_\infty, i}^{(r)}$ for every $0 \leq i \leq w$. For each i , choose an expression

$$F_i(\mathbf{z}) = \sum_{j=1}^{m_i} c_{i,j} E_r(\mathfrak{s}_{i,j}; \mathbf{z}),$$

where $c_{i,j} \in \mathbb{C}_\infty$ and $\text{wt}(\mathfrak{s}_{i,j}) = i$, and put

$$F_i^{(N)}(\mathbf{z}) := \sum_{j=1}^{m_i} c_{i,j} E_r^{(N)}(\mathfrak{s}_{i,j}; \mathbf{z}).$$

By Proposition A.6, as $N \rightarrow \infty$,

$$F_i^{(N)}(\mathbf{z}) \longrightarrow F_i(\mathbf{z}).$$

Evaluating the given relation at $\mathbf{z}_N$ and applying Proposition A.6 again, we obtain

$$0 = \sum_{i=0}^w F_i(\mathbf{z}_N) = \sum_{i=0}^w j_N(\mathbf{z})^i F_i^{(N)}(\mathbf{z}).$$

Dividing by $j_N(\mathbf{z})^w$ yields

$$F_w^{(N)}(\mathbf{z}) = - \sum_{i=0}^{w-1} j_N(\mathbf{z})^{i-w} F_i^{(N)}(\mathbf{z}).$$

Since $F_i^{(N)}(\mathbf{z}) \rightarrow F_i(\mathbf{z})$ as $N \rightarrow \infty$, the sequences $\{F_i^{(N)}(\mathbf{z})\}_{N \geq 1}$ are bounded. Moreover, for $N \gg 0$,

$$|j_N(\mathbf{z})|^{-1} = |\theta^N - z_{r-1}|^{-1} = q^{-N} \longrightarrow 0.$$

Hence, letting $N \rightarrow \infty$ in the preceding identity yields

$$F_w(\mathbf{z}) = 0.$$

Repeating the same argument for $F_0 + \cdots + F_{w-1} = 0$, we obtain

$$F_0 = F_1 = \cdots = F_w = 0.$$

Therefore,

$$\mathcal{Z}_{\mathbb{C}_\infty}^{(r)} = \bigoplus_{w \geq 0} \mathcal{Z}_{\mathbb{C}_\infty, w}^{(r)}.$$

□

A.2. Proof of Theorem A.1 (ii). Recall that in Remark 2.9, $\tilde{\pi}$ denotes a fixed fundamental period of the Carlitz A -module. For an index $\mathfrak{s}$, let

$$\tilde{\zeta}_A(\mathfrak{s}) = \tilde{\pi}^{-\text{wt}(\mathfrak{s})} \zeta_A(\mathfrak{s}), \quad \tilde{G}_2(\mathfrak{s}; \mathbf{z}) = \tilde{\pi}^{-\text{wt}(\mathfrak{s})} G_2(\mathfrak{s}; \mathbf{z}), \quad \tilde{E}_2(\mathfrak{s}; \mathbf{z}) = \tilde{\pi}^{-\text{wt}(\mathfrak{s})} E_2(\mathfrak{s}; \mathbf{z})$$

be the normalized MZV, MGS and MES of rank two. By Proposition 2.6, we have

$$(A.2.1) \quad \tilde{E}_2(\mathfrak{s}; \mathbf{z}) = \tilde{\zeta}_A(\mathfrak{s}) + \sum_{i=1}^{\text{dep}(\mathfrak{s})} \tilde{\zeta}_A(\mathfrak{s}^{(i)}) \tilde{G}_2(\mathfrak{s}_{(i)}; \mathbf{z}).$$

For any subfield L of $\mathbb{C}_\infty$ containing K and integer $w \geq 0$, we define

$$\tilde{\mathcal{Z}}_{L, w}^{(2)} := \text{span}_L \{ \tilde{E}_2(\mathfrak{s}; \mathbf{z}) \mid \text{wt}(\mathfrak{s}) = w \}, \quad \tilde{\mathcal{G}}_{L, \leq w}^{(2)} := \text{span}_L \{ \tilde{G}_2(\mathfrak{s}; \mathbf{z}) \mid \text{wt}(\mathfrak{s}) \leq w \}.$$

For $w \geq 0$, we further define

$$\text{gr}_w \tilde{\mathcal{G}}_L^{(2)} = \tilde{\mathcal{G}}_{L, \leq w}^{(2)} / \tilde{\mathcal{G}}_{L, \leq w-1}^{(2)}$$

where $\tilde{\mathcal{G}}_{L, \leq -1}^{(2)} := 0$. We note that we have a natural L -vector space isomorphism given by

$$(A.2.2) \quad \mathcal{Z}_{L, w}^{(2)} \xrightarrow{\sim} \tilde{\mathcal{Z}}_{L, w}^{(2)}, \quad E_2(\mathfrak{s}; \mathbf{z}) \mapsto \tilde{E}_2(\mathfrak{s}; \mathbf{z}).$$

Proposition A.7. *For every integer $w \geq 0$, the natural map*

$$\tilde{\mathcal{G}}_{K, \leq w}^{(2)} \otimes_K \mathbb{C}_\infty \xrightarrow{\sim} \tilde{\mathcal{G}}_{\mathbb{C}_\infty, \leq w}^{(2)}$$

is an isomorphism.

Proof. Notice that for any $\mathbf{z} \in \Omega^2$, $\Lambda_{\mathbf{z}'} = A$. Let $C = \phi^{\tilde{\pi}A}$ denote the Carlitz A -module and $\exp_C(X) = \exp_{\tilde{\pi}A}(X) \in K[[X]]$ denote the Carlitz exponential (see [Gos96, §3]). By Definition 2.2 and Proposition 2.3, we have

$$G_s^{\tilde{\pi}A}(X) \in K[X] \quad (s \geq 1), \quad \text{and} \quad G_s^{\tilde{\pi}A}(t_{\tilde{\pi}A}(\tilde{\pi}fz_1)) = \tilde{\pi}^{-s} G_s^A(t_A(fz_1)).$$

On the other hand, for every $f \in A_+$, we have

$$t_{\tilde{\pi}A}(\tilde{\pi}fz_1) = \frac{1}{\exp_{\tilde{\pi}A}(\tilde{\pi}fz_1)} = \frac{1}{C_f(\exp_{\tilde{\pi}A}(\tilde{\pi}z_1))} = \frac{1}{C_f(t_{\tilde{\pi}A}(\tilde{\pi}z_1)^{-1})} \in K[[t_{\tilde{\pi}A}(\tilde{\pi}z_1)]].$$

It follows that for every nonempty index $\mathfrak{s} = (s_1, \dots, s_m)$,

$$\tilde{G}_2(\mathfrak{s}; \mathbf{z}) = \sum_{\substack{f_1, \dots, f_m \in A_+ \\ \deg f_1 > \dots > \deg f_m}} G_{s_1}^{\tilde{\pi}A}(t_{\tilde{\pi}A}(\tilde{\pi}f_1z_1)) \cdots G_{s_m}^{\tilde{\pi}A}(t_{\tilde{\pi}A}(\tilde{\pi}f_mz_1)) \in K[[t_{\tilde{\pi}A}(\tilde{\pi}z_1)]].$$

Here, for each $M \geq 0$, (2.2.2) shows that only finitely many terms in the series above contribute to its $t_{\tilde{\pi}A}(\tilde{\pi}z_1)^M$ -coefficient. Hence this coefficient belongs to K . Therefore, we get an injective K -linear map

$$\tilde{\mathcal{G}}_{K, \leq w}^{(2)} \hookrightarrow K[[t_{\tilde{\pi}A}(\tilde{\pi}z_1)]].$$

Since $\mathbb{C}_\infty$ is flat over K ,

$$\tilde{\mathcal{G}}_{K, \leq w}^{(2)} \otimes_K \mathbb{C}_\infty \hookrightarrow \mathbb{C}_\infty[[t_{\tilde{\pi}A}(\tilde{\pi}z_1)]]$$

is still an injection. Notice that the image $\tilde{\mathcal{G}}_{K, \leq w}^{(2)} \otimes_K \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty[[t_{\tilde{\pi}A}(\tilde{\pi}z_1)]]$ is $\tilde{\mathcal{G}}_{\mathbb{C}_\infty, \leq w}$, so the assertion follows. $\square$

The following two consequences of Proposition A.7 are useful for the remaining proof:

(i) For $w \geq 1$, consider the exact sequence

$$0 \rightarrow \tilde{\mathcal{G}}_{K, \leq w-1}^{(2)} \rightarrow \tilde{\mathcal{G}}_{K, \leq w}^{(2)} \rightarrow \text{gr}_w \tilde{\mathcal{G}}_K^{(2)} \rightarrow 0.$$

By Proposition A.7, tensoring with $\mathbb{C}_\infty$ shows that the natural map

$$(A.2.3) \quad \text{gr}_w \tilde{\mathcal{G}}_K^{(2)} \otimes_K \mathbb{C}_\infty \xrightarrow{\sim} \text{gr}_w \tilde{\mathcal{G}}_{\mathbb{C}_\infty}^{(2)}$$

is an isomorphism.

(ii) For $w \geq 1$, we have

$$(A.2.4) \quad \tilde{\mathcal{G}}_{K, \leq w}^{(2)} \cap \tilde{\mathcal{G}}_{\mathbb{C}_\infty, \leq w-1}^{(2)} = \tilde{\mathcal{G}}_{K, \leq w-1}^{(2)}.$$

To see this, we first note that the inclusion " $\supseteq$ " is clear. On the other hand, suppose $x \in \tilde{\mathcal{G}}_{K, \leq w}^{(2)} \cap \tilde{\mathcal{G}}_{\mathbb{C}_\infty, \leq w-1}^{(2)}$. Since $\mathbb{C}_\infty/K$ is faithfully flat, the natural map

$$\mathrm{gr}_w \tilde{\mathcal{G}}_K^{(2)} \rightarrow \mathrm{gr}_w \tilde{\mathcal{G}}_K^{(2)} \otimes_K \mathbb{C}_\infty$$

is an injection. Composing with (A.2.3), we obtain an injection

$$\mathrm{gr}_w \tilde{\mathcal{G}}_K^{(2)} \hookrightarrow \mathrm{gr}_w \tilde{\mathcal{G}}_K^{(2)} \otimes_K \mathbb{C}_\infty \xrightarrow{\sim} \mathrm{gr}_w \tilde{\mathcal{G}}_{\mathbb{C}_\infty}^{(2)}$$

We see that the image of the element $x + \tilde{\mathcal{G}}_{K, \leq w-1}^{(2)} \in \mathrm{gr}_w \tilde{\mathcal{G}}_K^{(2)}$ under the composite map above is

$$x + \tilde{\mathcal{G}}_{\mathbb{C}_\infty, \leq w-1}^{(2)} = 0 \in \mathrm{gr}_w \tilde{\mathcal{G}}_{\mathbb{C}_\infty}^{(2)}$$

and thus $x + \tilde{\mathcal{G}}_{K, \leq w-1}^{(2)} = 0 \in \mathrm{gr}_w \tilde{\mathcal{G}}_K^{(2)}$, which gives the desired result.

Proposition A.8. *Let $L = K$ or $\mathbb{C}_\infty$. Then for any integer $w \geq 0$ the map*

$$\varphi_{L,w} := \left(\tilde{E}_2(\mathfrak{s}; \mathbf{z}) \mapsto \tilde{G}_2(\mathfrak{s}; \mathbf{z}) \mod \tilde{\mathcal{G}}_{L, \leq w-1}^{(2)} \right) : \tilde{\mathcal{Z}}_{L,w}^{(2)} \rightarrow \mathrm{gr}_w \tilde{\mathcal{G}}_L^{(2)}.$$

is a well-defined isomorphism.

Proof. Assume $w \geq 1$ as the case $w = 0$ is clear. We first show the well-definedness for $L = \mathbb{C}_\infty$. Suppose

$$0 = \sum_{i=1}^m c_i \tilde{E}_2(\mathfrak{s}_i; \mathbf{z}) \in \tilde{\mathcal{Z}}_{\mathbb{C}_\infty, w}^{(2)}$$

for some $c_i \in \mathbb{C}_\infty$. Then by (A.2.1) we have

$$\sum_{i=1}^m c_i \tilde{G}_2(\mathfrak{s}_i; \mathbf{z}) = - \sum_{i=1}^m c_i \sum_{j=0}^{\mathrm{dep}(\mathfrak{s}_i)-1} \tilde{\zeta}_A((\mathfrak{s}_i)^{(j)}) \tilde{G}_2((\mathfrak{s}_i)_{(j)}) \in \tilde{\mathcal{G}}_{\mathbb{C}_\infty, \leq w-1}^{(2)}.$$

Hence, $\varphi_{\mathbb{C}_\infty, w}$ is well-defined.

Now, we show the well-definedness for $L = K$. Suppose that

$$0 = \sum_{i=1}^m c_i \tilde{E}_2(\mathfrak{s}_i; \mathbf{z}) \in \tilde{\mathcal{Z}}_{K, w}^{(2)}$$

for some $c_i \in K$. Similarly, by (A.2.1) and (A.2.4), we see that

$$\sum_{i=1}^m c_i \tilde{G}_2(\mathfrak{s}_i; \mathbf{z}) \in \tilde{\mathcal{G}}_{K, \leq w}^{(2)} \cap \tilde{\mathcal{G}}_{\mathbb{C}_\infty, \leq w-1}^{(2)} = \tilde{\mathcal{G}}_{K, \leq w-1}^{(2)}$$

Hence, $\varphi_{K, w}$ is well-defined.

For $L = K$ or $L = \mathbb{C}_\infty$, we note that the surjectivity of $\varphi_{L, w}$ is clear from the definition. We then show the injectivity of $\varphi_{L, w}$ as follows. Suppose that $\sum_{i=1}^m c_i \tilde{E}_2(\mathfrak{s}_i; \mathbf{z}) \in \tilde{\mathcal{Z}}_{L, w}^{(2)}$ such that

$$\sum_{i=1}^m c_i \tilde{G}_2(\mathfrak{s}_i; \mathbf{z}) \in \tilde{\mathcal{G}}_{L, \leq w-1}^{(2)} \subseteq \tilde{\mathcal{G}}_{\mathbb{C}_\infty, \leq w-1}^{(2)}.$$

By (A.2.1), we have

$$\sum_{i=1}^m c_i \tilde{E}_2(\mathfrak{s}_i; \mathbf{z}) - \sum_{i=1}^m c_i \tilde{G}_2(\mathfrak{s}_i; \mathbf{z}) \in \tilde{\mathcal{G}}_{\mathbf{C}_\infty, \leq w-1}^{(2)},$$

which implies

$$\sum_{i=1}^m c_i \tilde{E}_2(\mathfrak{s}_i; \mathbf{z}) \in \tilde{\mathcal{G}}_{\mathbf{C}_\infty, \leq w-1}^{(2)} \subseteq \sum_{j=0}^{w-1} \tilde{\mathcal{Z}}_{\mathbf{C}_\infty, j}^{(2)}.$$

Therefore, by Theorem A.1 (i), we obtain

$$\sum_{i=1}^m c_i E_2(\mathfrak{s}_i; \mathbf{z}) \in \sum_{j=0}^{w-1} \mathcal{Z}_{\mathbf{C}_\infty, j}^{(2)} \cap \mathcal{Z}_{\mathbf{C}_\infty, w}^{(2)} = 0.$$

This proves the injectivity. $\square$

Proof of Theorem A.1 (ii). It suffices to show the case $r = 2$. For general $r \geq 2$, the result follows by adapting the induction argument of Theorem 1.5 (ii), with $\bar{K}$ replaced by $\mathbf{C}_\infty$ and the base case of the induction replaced by the case $r = 2$.

Assume $r = 2$ and $w \geq 1$. The map $\tilde{\mathcal{Z}}_{K, w}^{(2)} \otimes_K \mathbf{C}_\infty \rightarrow \tilde{\mathcal{Z}}_{\mathbf{C}_\infty, w}^{(2)}$ fits into the following commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{Z}}_{K, w}^{(2)} \otimes_K \mathbf{C}_\infty & \longrightarrow & \tilde{\mathcal{Z}}_{\mathbf{C}_\infty, w}^{(2)} \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{gr}_w \tilde{\mathcal{G}}_K^{(2)} \otimes_K \mathbf{C}_\infty & \xrightarrow{\sim} & \mathrm{gr}_w \tilde{\mathcal{G}}_{\mathbf{C}_\infty}^{(2)} \end{array}$$

where the three isomorphisms follow from Proposition A.8 and (A.2.3). This implies $\tilde{\mathcal{Z}}_{K, w}^{(2)} \otimes_K \mathbf{C}_\infty \xrightarrow{\sim} \tilde{\mathcal{Z}}_{\mathbf{C}_\infty, w}^{(2)}$ is an isomorphism. Therefore, by (A.2.2), we obtain

$$\mathcal{Z}_{K, w}^{(2)} \otimes_K \mathbf{C}_\infty \xrightarrow{\sim} \mathcal{Z}_{\mathbf{C}_\infty, w}^{(2)}.$$

The assertion for $r = 2$ follows from summing over all $w \geq 0$. This completes the proof. $\square$

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