UNIQUENESS OF v-ADIC GAMMA FUNCTIONS IN GROSS-KOBLITZ-TYPE FORMULAS OVER FUNCTION FIELDS

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Abstract. Inspired by Adolphson's pioneering work on the uniqueness of the p-adic gamma function in the Gross-Koblitz formula, we study an analogous question in a more general setting in this paper. Our main theorem simultaneously deals with the uniqueness problems for the classical p-adic gamma function and various v-adic gamma functions in positive characteristic in their respective Gross-Koblitz-type formulas.

1. Introduction

1.1. **Motivation.** We recall that the classical Euler's gamma function is given by

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt, \quad \mathrm{Re}(z) > 0,$$

which has meromorphic continuation to the complex plane and interpolates the factorials as one has $\Gamma(n+1) = n!$ for all natural numbers n. The special gamma values are referred to the values of Γ at proper fractions, and they are interesting because of having connection with periods. For instance, it is known by Lerch and Chowla-Selberg (see [12]) that periods of elliptic curves with complex multiplication over number fields can be expressed as products of special gamma values. A folklore conjecture asserts that all these special gamma values are transcendental numbers. To date, the transcendence of $\Gamma(z)$ is known only when the denominator of $z \in \mathbb{Q} \setminus \mathbb{Z}$ is either $2, 4$ or 6. For example, one knows that $\Gamma(1/2) = \sqrt{\pi}$

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is transcendental over $\mathbb{Q}$. The latter two cases mentioned above are consequences of Chudnovsky's result [7] on the algebraic independence of a non-zero period and the associated quasi-period of a CM elliptic curve defined over $\overline{\mathbb{Q}}$.

Let p be a prime number. Morita [11] studied the p-adic interpolation problem of the factorials, and then successfully defined the p-adic gamma function Γ_p, which is the unique continuous function on the p-adic integers $\mathbb{Z}_p$ such that

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1 \leq t \leq n-1 \\ p \nmid t}} t, \quad \forall\, n \in \mathbb{N}.$$

The values $\{\Gamma_p(z) : z \in (\mathbb{Q} \cap \mathbb{Z}_p) \setminus \mathbb{Z}\}$ are called special p-adic gamma values. Compared with the (conjectural) situation of Euler's gamma, a peculiar phenomenon in the p-adic world occurs: A class of special p-adic gamma values turns out to be algebraic numbers when $p > 2$. This result is a consequence of the celebrated *Gross-Koblitz formula* in [10], which is reviewed as follows.

We let p be an odd prime and $q := p^\ell$ for some $\ell \in \mathbb{N}$. Consider the $(q-1)$-st cyclotomic field $K := \mathbb{Q}(\mu_{q-1})$. We choose a prime $\mathfrak{P}$ of K above p and let $\mathbb{F}_{\mathfrak{P}}$ be its residue field (which is a finite field of q elements). We let χ be the group homomorphism from $\mathbb{F}_{\mathfrak{P}}^\times$ to μ_{q-1} in $\overline{\mathbb{Q}}_p^\times$ which is the inverse of reduction map (the Teichmüller character), and choose an isomorphism $\psi : \mathbb{Z}/p\mathbb{Z} \to \mu_p$ as abelian groups. For $x \in (q-1)^{-1}\mathbb{Z}$ with $0 \leq x < 1$, we consider the Gauss sum

$$g_\ell(x) := -\sum_{z \in \mathbb{F}_{\mathfrak{P}}^\times} \chi(z^{-x(q-1)})\psi(\mathrm{Tr}_{\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_p}(z)).$$

Let $\varpi_p \in \mathbb{Q}_p(\psi(1))$ be the unique $(p-1)$-st root of $-p$ for which $\varpi_p \equiv \psi(1) - 1 \pmod{(\psi(1)-1)^2}$.

Theorem 1.1 (Gross-Koblitz formula)**.** *For $x \in (q-1)^{-1}\mathbb{Z}$ with $0 \leq x < 1$, one has*

$$g_\ell(x) = \varpi_p^{(p-1)\sum_{j=0}^{\ell-1}\langle p^j x\rangle} \prod_{j=0}^{\ell-1} \Gamma_p\left(\langle p^j x\rangle\right). \tag{1}$$

Here, $\langle y\rangle := y - [y]$ is the fractional part of a non-negative real number y.

In particular, the Gross-Koblitz formula together with the translation formula of Γ_p (see [11, p. 256]) implies that $\Gamma_p(r/(p-1))$ is algebraic over $\mathbb{Q}$ for all $r \in \mathbb{Z}$.

After the above formula was established, Greenberg asked a question concerning the uniqueness of p-adic gamma function in Gross-Koblitz formula: Is there any other continuous function H from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ such that (1) remains true when Γ_p is replaced by H? The answer to this problem was provided by Adolphson in [1] where a complete classification of such H was given. Specifically, we consider the function $\varphi : \mathbb{Z}_p \to \mathbb{Z}_p$ defined by

$$\varphi\left(\sum_{i=0}^{\infty} x_i p^i\right) := \sum_{i=0}^{\infty} x_{i+1} p^i \tag{2}$$

where $0 \leq x_i < p$ for all i. Then he showed (see [1, pp. 58–59])

Theorem 1.2 (Adolphson). *A function $H : \mathbb{Z}_p \to \mathbb{Q}_p$ is continuous non-vanishing satisfying (1) if and only if*

$$H(x) = \Gamma_p(x) \cdot \frac{G(x)}{G\left(-\varphi(-x)\right)}$$

on $\mathbb{Z}_p$, where $G : \mathbb{Z}_p \to \mathbb{Q}_p$ is any continuous non-vanishing function.

Adolphson's result motivates us to study the positive characteristic counterparts in the function field setting.

1.2. **v-adic Gamma Functions.** We now turn our focus to the function field setting. Let p be any prime and q be a power of p. Let $A := \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over a finite field $\mathbb{F}_q$ of q elements, and $k := \mathbb{F}_q(\theta)$ be its field of fractions. Let A_+ be the set of all monic polynomials in A and $A_{+,i}$ (for $i \geq 0$) be its subset consisting of all monic polynomials of degree i. We fix an irreducible $v \in A_{+,d}$ and let k_v be the completion of k at v with ring of integers A_v. We let $\overline{k}$ be the algebraic closure of k contained in a fixed algebraic closure $\overline{k}_v$ of k_v. For an element $x \in A_v$, put

$$x^\flat := \begin{cases} x, & \text{if } x \in A_v^\times, \\ 1, & \text{if } x \in vA_v. \end{cases}$$

Definition 1.3. (1) (*v-adic arithmetic gamma function*, [8, Appendix]): For $y = \sum_{i=0}^\infty y_i q^i \in \mathbb{Z}_p$ with $0 \leq y_i < q$ for all i, define $\Gamma_v^{\mathrm{ari}} : \mathbb{Z}_p \to A_v^\times$ by

$$\Gamma_v^{\mathrm{ari}}(y+1) := \prod_{i=0}^\infty \left(- \prod_{a \in A_{+,i}} a^\flat \right)^{y_i}.$$

(2) (*v-adic geometric gamma function*, [14, Section 5]): Define $\Gamma_v^{\mathrm{geo}} : A_v \to A_v^\times$ by

$$\Gamma_v^{\mathrm{geo}}(x) := \frac{1}{x^\flat} \prod_{i=0}^\infty \left(\prod_{a \in A_{+,i}} \frac{a^\flat}{(x+a)^\flat} \right).$$

(3) (*v-adic two-variable gamma function*, [9, Subsection 9.9]): For $(x,y) \in A_v \times \mathbb{Z}_p$ with y as in the arithmetic case, define $\Gamma_v^{\mathrm{two}} : A_v \times \mathbb{Z}_p \to A_v^\times$ by

$$\Gamma_v^{\mathrm{two}}(x, y+1) := \frac{1}{x^\flat} \prod_{i=0}^\infty \left(\prod_{a \in A_{+,i}} \frac{a^\flat}{(x+a)^\flat} \right)^{y_i}.$$

The arithmetic case is the Morita-style v-adic interpolation of Carlitz factorials [3] $\Gamma^{\mathrm{ari}} : \mathbb{N} \to A$ given by

$$\Gamma^{\mathrm{ari}}\left(1 + \sum_{i=0}^n y_i q^i\right) := \prod_{i=0}^n \left(\prod_{a \in A_{+,i}} a \right)^{y_i} \quad \text{where} \quad 0 \leq y_i < q.$$

This is regarded as an analog of $\Gamma(y+1) = y!$ for $y \in \mathbb{N}$ in view of W. Sinnott's result that (see [9, Theorem 9.1.1])

$$\mathrm{ord}_f(\Gamma^{\mathrm{ari}}(y+1)) = \sum_{e \geq 1} \left\lfloor \frac{y}{q^{e \deg f}} \right\rfloor$$

for any monic prime f. On the other hand, the geometric case arises from the Morita-style v-adic interpolation of the function $\Gamma^{\mathrm{geo}} : A \to k \cup \{\infty\}$ given by

$$\Gamma^{\mathrm{geo}}(x) := \frac{1}{x} \prod_{a \in A_+} \frac{a}{x+a} = x^{-1} \prod_{a \in A_+} \left(1 + \frac{x}{a}\right)^{-1}.$$

Note that Γ^{geo} has simple poles at $-A_+ \cup \{0\} := \{-a \mid a \in A_+\} \cup \{0\}$, analogous to the case that Euler's Γ-function has simple poles at $-\mathbb{N} \cup \{0\} := \{-n \mid n \in \mathbb{N}\} \cup \{0\}$. Finally, for the two-variable case, one sees that

$$\Gamma_v^{\mathrm{two}}\left(x, 1 - \frac{1}{q-1}\right) = \Gamma_v^{\mathrm{geo}}(x).$$

Note that all three gamma functions in Definition 1.3 are continuous and non-vanishing as they are defined via the interpolations of continuous functions and take values in the units of A_v.

1.3. **Gross-Koblitz-Type Formulas for v-adic Gamma Functions.** There are analogs of Gross-Koblitz formulas for these v-adic gamma functions by the work of Thakur [13] and Chang [6]. Before stating the precise formulas we prepare some necessary background (see [9, Chapter 3] for more details). Let K be a field together with an $\mathbb{F}_q$-algebra homomorphism $\iota : A \to K$. Let $\tau \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a)$ be the q-th power Frobenius morphism on the additive group $\mathbb{G}_a$ over K. We let $K\{\tau\}$ be the twisted polynomial ring in τ over K with usual addition but the multiplication is given by $\tau x = x^q \tau$ for all $x \in K$. Recall that the Carlitz module over K is the $\mathbb{F}_q$-algebra homomorphism $C : A \to K\{\tau\}$ defined by $C_\theta := \iota(\theta) + \tau$. It induces a new A-module structure $C(K)$ on K given by $a \cdot x := C_a(x)$ for all $a \in A$ and $x \in K$. In the case $K = \overline{k}$ (with the natural structure map), we let $\Lambda_a := C(\overline{k})[a]$ be the a-torsion points of $C(\overline{k})$ for any $a \in A$. Then it is a fact that Λ_a is isomorphic to A/a as A-modules.

To state the analogs of Gross-Koblitz formulas for three v-adic gamma functions, we define the notions of fractional part on the rational number field $\mathbb{Q}$ and function field k. Note for $y \in \mathbb{Q}$, its fractional part $\langle y \rangle$ can be defined as the unique element in $\mathbb{Q}$ such that $0 \leq \langle y \rangle < 1$ and $y \equiv \langle y \rangle \pmod{\mathbb{Z}}$. On the other hand, one recalls the normalized ∞-adic absolute value on k is given by $|0|_\infty := 0$ and $|f/g|_\infty := q^{\deg f - \deg g}$ for any $f, g \in A \setminus \{0\}$. (In what follows, we will omit the subscript $|\cdot|_\infty$ and denote it as $|\cdot|$ for convenience.) Thus for $x \in k$, we define its fractional part $\langle x \rangle$ as the unique element in k such that $0 \leq |\langle x \rangle| < 1$ and $x \equiv \langle x \rangle \pmod{A}$.

First, we consider the arithmetic case [13]. Let $\chi : A/v \to \mathbb{F}_{q^d} \subseteq \overline{k}$ be the Teichmüller character, and choose an A-module isomorphism $\psi : A/v \to \Lambda_v$. Then Thakur defined the *arithmetic Gauss sum* to be

$$g^{\mathrm{ari}} := - \sum_{z \in (A/v)^\times} \chi(z^{-1}) \psi(z) \in k(\Lambda_v)\mathbb{F}_{q^d}.$$

(The same element is denoted as g_0 in the original paper.) More generally, take any natural number ℓ. For $y \in (q^{d\ell}-1)^{-1}\mathbb{Z}$ with $0 \leq y < 1$, we write

$$y = \sum_{s=0}^{d\ell-1} \frac{y_s q^s}{q^{d\ell}-1} \quad (0 \leq y_s < q \text{ for all } s).$$

We let $\tau_q \in \mathrm{Gal}(k\mathbb{F}_{q^d}/k)$ be the q-th power Frobenius map on the constant field, and extend it canonically to $\mathrm{Gal}(k(\Lambda_v)\mathbb{F}_{q^d}/k) \simeq \mathrm{Gal}(k(\Lambda_v)/k) \times \mathrm{Gal}(k\mathbb{F}_{q^d}/k)$. Then we consider the integral group ring element $\sum_{s=0}^{d\ell-1} y_s\tau_q^s \in \mathbb{Z}[\mathrm{Gal}(k(\Lambda_v)\mathbb{F}_{q^d}/k)]$ acting on the arithmetic Gauss sum g^{ari}, and define

$$G_\ell^{\mathrm{ari}}(y) := (-1)^{\ell(d-1)} \prod_{s=0}^{d\ell-1} (g^{\mathrm{ari}})^{y_s\tau_q^s}.$$

Let $\varpi_v \in k_v(\psi(1))$ be the unique (q^d-1)-st root of $-v$ for which $\varpi_v \equiv -\psi(1) \pmod{\psi(1)^2}$. Then under this setting, Thakur proved the following analog of Gross-Koblitz formula for v-adic arithmetic gamma function (see [13, Theorem VI] and [14, Theorem 4.8]).

Theorem 1.4. *For $y \in (q^{d\ell}-1)^{-1}\mathbb{Z}$ with $0 \leq y < 1$, one has*

$$G_\ell^{\mathrm{ari}}(y) = \varpi_v^{(q^d-1)\sum_{j=0}^{\ell-1}\langle q^{dj}y\rangle} \prod_{j=0}^{\ell-1} \Gamma_v^{\mathrm{ari}}\left(\langle q^{dj}y\rangle\right). \tag{3}$$

Next, we consider the geometric and two-variable cases [6]. Fix a positive integer ℓ and let $\mathfrak{n} := v^\ell$. Consider the Carlitz $(\mathfrak{n}-1)$-st cyclotomic function field $K := k(\Lambda_{\mathfrak{n}-1})$. We let $\mathfrak{P}$ be the prime in K above v corresponding to the inclusions $K \subseteq \overline{k} \subseteq \overline{k}_v$, and let $\mathbb{F}_{\mathfrak{P}}$ be the residue field (which is a finite field of $q^{d\ell}$ elements). Then via the natural structure map on $\mathbb{F}_{\mathfrak{P}}$, it obtains a new A-module structure $C(\mathbb{F}_{\mathfrak{P}})$ via the Carlitz module. We let $\omega : C(\mathbb{F}_{\mathfrak{P}}) \to \Lambda_{\mathfrak{n}-1}$ be the A-module isomorphism which is the inverse of reduction map, and let $\psi : \mathbb{F}_{\mathfrak{P}} \to \mathbb{F}_{q^{d\ell}} \subseteq \overline{k}$ be the usual Teichmüller embedding. Then for $x \in (\mathfrak{n}-1)^{-1}A$ with $|x| < 1$, we define the *geometric Gauss sum* to be

$$g_x^{\mathrm{geo}} := 1 + \sum_{z\in\mathbb{F}_{\mathfrak{P}}^\times} \omega\left(C_{x(\mathfrak{n}-1)}(z^{-1})\right)\psi(z) \in K\mathbb{F}_{q^{d\ell}}.$$

More generally, we let $\tau_q \in \mathrm{Gal}(k\mathbb{F}_{q^{d\ell}}/k)$ be the q-th power Frobenius map on the constant field, and extend it canonically to $\mathrm{Gal}(K\mathbb{F}_{q^{d\ell}}/k) \simeq \mathrm{Gal}(K/k) \times \mathrm{Gal}(k\mathbb{F}_{q^{d\ell}}/k)$. For $y \in (q^{d\ell}-1)^{-1}\mathbb{Z}$ with $0 \leq y < 1$ written as in the arithmetic case, we consider the integral group ring element $\sum_{s=0}^{d\ell-1} y_s\tau_q^s \in \mathbb{Z}[\mathrm{Gal}(K\mathbb{F}_{q^{d\ell}}/k)]$ acting on the geometric Gauss sum g_x^{geo}, and define

$$G_\ell^{\mathrm{geo}}(x,y) := \prod_{s=0}^{d\ell-1} (g_x^{\mathrm{geo}})^{y_s\tau_q^s}.$$

In particular, put

$$G_\ell^{\mathrm{geo}}(x) := G_\ell^{\mathrm{geo}}\left(x, \frac{1}{q-1}\right) = \prod_{s=0}^{d\ell-1} (g_x^{\mathrm{geo}})^{\tau_q^s}.$$

Then under this setting, we have the following analogs of Gross-Koblitz formulas for v-adic geometric and two-variable gamma functions [6, Theorem 5.1.3].

Theorem 1.5. *Suppose $x \in (\mathfrak{n}-1)^{-1}A$ and $y \in (q^{d\ell}-1)^{-1}\mathbb{Z}$ with $|x|<1$ and $0 \le y < 1$, write*

$$x = \sum_{j=0}^{\ell-1} \frac{x_j v^j}{v^\ell - 1} \quad (\deg x_j < d \text{ for all } j), \quad y = \sum_{s=0}^{d\ell-1} \frac{y_s q^s}{q^{d\ell}-1} \quad (0 \le y_s < q \text{ for all } s).$$

Then we have

$$G_\ell^{\mathrm{geo}}(x) = \prod_{j=0}^{\ell-1} \frac{\delta_{x,j}}{\langle v^j x\rangle^\flat} \cdot \prod_{j=0}^{\ell-1} \Gamma_v^{\mathrm{geo}}\left(\langle v^j x\rangle\right)^{-1} \tag{4}$$

and

$$G_\ell^{\mathrm{geo}}(x,y) = \left(\prod_{j=0}^{\ell-1} \frac{\langle v^j x\rangle^\flat}{\delta_{x,j}^{q-1-y_{dj+\deg x_j}}}\right) G_\ell^{\mathrm{geo}}(x)^{q-1} \prod_{j=0}^{\ell-1} \Gamma_v^{\mathrm{two}}\left(\langle v^j x\rangle, \langle q^{dj} y\rangle\right) \tag{5}$$

where $\delta_{x,j} := v\langle v^{\ell-j-1}x\rangle$ if $x_j \in A_+$ and 1 otherwise.

Remark 1.6. Note that the translation formulas of these v-adic gamma functions imply that

$$\frac{\Gamma_v^{\mathrm{ari}}(y+r)}{\Gamma_v^{\mathrm{ari}}(y)}, \frac{\Gamma_v^{\mathrm{geo}}(x+a)}{\Gamma_v^{\mathrm{geo}}(x)}, \frac{\Gamma_v^{\mathrm{two}}(x+a, y+r)}{\Gamma_v^{\mathrm{two}}(x,y)} \in k^\times, \quad \forall\, r \in \mathbb{Z}, a \in A.$$

Hence, similar to the classical case, the Gross-Koblitz-type formulas (3),(4),(5) imply that $\Gamma_v^{\mathrm{ari}}(r/(q^d-1)), \Gamma_v^{\mathrm{geo}}(a/(v-1)), \Gamma_v^{\mathrm{two}}(a/(v-1), r/(q^d-1))$ are algebraic over k for all $r \in \mathbb{Z}$ and $a \in A$. We further mention that in the paper [5], Chang, Wei and Yu proved the transcendence of those special v-adic arithmetic gamma values which are not from Thakur's Gross-Koblitz formula. That is, for $a/b \in \mathbb{Q}^\times \cap \mathbb{Z}_p$ with $\gcd(a,b)=1$, we have $\Gamma_v^{\mathrm{ari}}(a/b)$ is transcendental over k if and only if $b \nmid q^d - 1$.

We also remark that there are notions of ∞-adic gamma functions in positive characteristic, and the transcendence theory for these functions is well understood, as established in [4], [2], and [16].

1.4. **Main Theorem.** In this paper, we consider the analogous question of Greenberg for these three v-adic gamma functions: For each case, is there any other non-vanishing continuous k_v-valued function H defined on the same domain as the respective gamma function, such that the corresponding Gross-Koblitz-type formula ((3),(4),(5)) remains true when the gamma is replaced by H? For this problem, we give a precise description of all such functions parallel to the classical result.

To treat these three cases at once, we consider a more general setting. Let $\mathcal{O}$ be a direct product of finitely many rings $\mathcal{O}_t$ (indexed by t), where each $\mathcal{O}_t$ is either $\mathbb{Z}$ or A. We let $\overline{\mathcal{O}}_t$ be the completion $\mathbb{Z}_{p_t}$ or A_{v_t} (depending on the choice of $\mathcal{O}_t$) at a fixed finite prime p_t or v_t, and let $\overline{\mathcal{O}}$ be the direct product of $\overline{\mathcal{O}}_t$.

For each t, we fix a power of the given uniformizer $\pi_t \in \overline{\mathcal{O}}_t$ and put $\boldsymbol{\pi} := (\pi_t)_t \in \overline{\mathcal{O}}$. Let $\mathbf{1} := (1, \dots, 1) \in \overline{\mathcal{O}}$ be the identity element. For $\mathbf{x} = (\mathbf{x}_t)_t \in \overline{\mathcal{O}}$, we write

$$\mathbf{x}_t := \sum_{i=0}^{\infty} x_{t,i} \pi_t^i \in \overline{\mathcal{O}}_t$$

in π_t-adic expansion. This means that $x_{t,i}$ is an integer with $0 \leq x_{t,i} < \pi_t$ when $\mathcal{O}_t = \mathbb{Z}$ or a polynomial in A with $|x_{t,i}| < |\pi_t|$ (i.e., $\deg x_{t,i} < \deg \pi_t$) when $\mathcal{O}_t = A$. Define $\varphi : \overline{\mathcal{O}} \to \overline{\mathcal{O}}$ by $\varphi(\mathbf{x}) = (\varphi_t(\mathbf{x}_t))_t$ where each $\varphi_t : \overline{\mathcal{O}}_t \to \overline{\mathcal{O}}_t$ is given by (cf. (2))

$$\varphi_t(\mathbf{x}_t) := \sum_{i=0}^{\infty} x_{t,i+1}\pi_t^i.$$

We also define the fractional part of elements in the product of fraction fields of $\mathcal{O}_t$, denoted as $\prod \mathrm{Frac}(\mathcal{O}_t)$, by taking the fractional parts componentwise. That is, define $\langle \mathbf{x} \rangle := (\langle \mathbf{x}_t \rangle)_t$ for any $\mathbf{x} \in \prod \mathrm{Frac}(\mathcal{O}_t)$.

Our main result in this paper is to study and answer analogous problems of Greenberg in the setting as general as possible. For any $n \in \mathbb{N}$ and $\mathbf{x} = (\mathbf{x}_t)_t \in (\boldsymbol{\pi}^n - \mathbf{1})^{-1}\mathcal{O}$, we set up the condition:

$$\begin{cases} 0 \leq \mathbf{x}_t < 1, & \text{if } \mathcal{O}_t = \mathbb{Z}, \\ |\mathbf{x}_t| < 1, & \text{if } \mathcal{O}_t = A. \end{cases} \tag{6}$$

Our main theorem is stated as follows.

Theorem 1.7. *Given any complete non-Archimedean field K, let $\boldsymbol{\Gamma} : \overline{\mathcal{O}} \to K$ be a continuous non-vanishing function. Then $H : \overline{\mathcal{O}} \to K$ is a function satisfying the following two conditions*

- *H is continuous and non-vanishing;*
- *For all $n \in \mathbb{N}$ and all $\mathbf{x} \in (\boldsymbol{\pi}^n - \mathbf{1})^{-1}\mathcal{O}$ with condition (6), we have*

$$\prod_{j=0}^{n-1} \boldsymbol{\Gamma}\left(\langle \boldsymbol{\pi}^j \mathbf{x} \rangle\right) = \prod_{j=0}^{n-1} H\left(\langle \boldsymbol{\pi}^j \mathbf{x} \rangle\right); \tag{7}$$

if and only if

$$H(\mathbf{x}) = \boldsymbol{\Gamma}(\mathbf{x}) \cdot \frac{G(\mathbf{x})}{G\left(-\varphi(-\mathbf{x})\right)}$$

on $\overline{\mathcal{O}}$, where $G : \overline{\mathcal{O}} \to K$ is any continuous non-vanishing function.

As corollaries, we obtain the complete characterizations of all continuous non-vanishing functions satisfying the Gross-Koblitz-type formulas for the v-adic arithmetic, geometric and two-variable gamma functions, respectively. Specifically, for each $\bullet$ = ari, geo, two, we choose $(\overline{\mathcal{O}}^\bullet, \boldsymbol{\pi}^\bullet)$ to be $(\mathbb{Z}_p, q^d), (A_v, v), (A_v \times \mathbb{Z}_p, (v, q^d))$, respectively. We denote $\varphi^\bullet : \overline{\mathcal{O}}^\bullet \to \overline{\mathcal{O}}^\bullet$ to be the corresponding φ function. Then we have the following consequence.

Corollary 1.8. *For any $\bullet \in \{\mathrm{ari}, \mathrm{geo}, \mathrm{two}\}$, a function $H : \overline{\mathcal{O}}^\bullet \to k_v$ is continuous non-vanishing and satisfies the corresponding Gross-Koblitz-type formula ((3), (4), (5)) if and only if*

$$H(\mathbf{x}) = \Gamma_v^\bullet(\mathbf{x}) \cdot \frac{G(\mathbf{x})}{G\left(-\varphi^\bullet(-\mathbf{x})\right)}$$

on $\overline{\mathcal{O}}^\bullet$, where $G : \overline{\mathcal{O}}^\bullet \to k_v$ is any continuous non-vanishing function.

As additional corollaries, Theorem 1.7 not only addresses the classical case (see Theorem 1.2), but also provides similar characterizations for the v-adic arithmetic gamma functions for "general A" in their Gross-Koblitz-type formulas due to Thakur [15].

Following Thakur's setting in [14, 2], we provide another class of examples where G are given explicitly.

Example 1.9. Given any complete non-Archimedean field K, let $f : \mathbb{Z}_p \to K$ be a continuous non-vanishing function given by $f(\sum_{i=0}^{\infty} a_i q^i) := \prod_{i=0}^{\infty} f_i^{a_i}$, where $0 \leq a_i < q$ and $f_i \in K$ for all $i \geq 0$ such that f converges everywhere on $\mathbb{Z}_p$, and $g(x) := f(x-1)$. We note that this setting includes not only $\Gamma_v^{\mathrm{ari}}(\cdot)$ and $\Gamma_v^{\mathrm{two}}(x, \cdot)$ in Definition 1.3, but also various arithmetic gamma functions for "general A" in both ∞-adic and v-adic sides [14].

Fix any $\ell \in \mathbb{N}$ and take $(\overline{\mathcal{O}}, \boldsymbol{\pi}) = (\mathbb{Z}_p, q^\ell)$. Put $f'(x) := f(\varphi(x))$ and $g'(x) := f'(x-1)$. Note that in this case, $\varphi : \mathbb{Z}_p \to \mathbb{Z}_p$ is given by

$$\varphi\left(\sum_{i=0}^{\infty} a_i q^{\ell i}\right) := \sum_{i=0}^{\infty} a_{i+1} q^{\ell i}$$

where $0 \leq a_i < q^\ell$ for all $i \geq 0$. One checks that for all $n \in \mathbb{N}$ and $x \in (q^{n\ell}-1)^{-1}\mathbb{Z}$ with $0 \leq x < 1$, if we write

$$x = \sum_{i=0}^{n\ell-1} \frac{x_i q^i}{q^{n\ell}-1} \quad (0 \leq x_i < q \text{ for all } i),$$

then

$$\prod_{j=0}^{n-1} g'\left(\langle q^{\ell j} x\rangle\right) = \prod_{j=0}^{n-1} g\left(\langle q^{\ell j} x\rangle\right) = \prod_{i=0}^{\ell-1} g\left(1 - \frac{q^i}{q^\ell - 1}\right)^{\sum_{j=0}^{n-1}(q-1-x_{i+j\ell})}.$$

In this case, we have $G(x) = g(x)$ and

$$g(x) = g'(x) \cdot \frac{g(x)}{g(-\varphi(-x))}.$$

1.5. **Strategy of Proof.** Our proof follows the original paper [1] closely (in particular, the Remark 3 and the theorem after it). First, it is observed in Section 2.1 that (7) can be rewritten as (for the same conditions on $\mathbf{x}$)

$$\prod_{j=0}^{n-1} \boldsymbol{\Gamma}\left(-\varphi^{(j)}(-\mathbf{x})\right) = \prod_{j=0}^{n-1} H\left(-\varphi^{(j)}(-\mathbf{x})\right).$$

Such expression will immediately imply the "if" direction.

For the "only if" part, we set $F(\mathbf{x}) := H(\mathbf{x})/\boldsymbol{\Gamma}(\mathbf{x})$. Then what we want to show is that F must be of the form (Theorem 2.1 and [1, Theorem 1])

$$F(\mathbf{x}) = \frac{G(\mathbf{x})}{G\left(-\varphi(-\mathbf{x})\right)}$$

on $\overline{\mathcal{O}}$ for some continuous non-vanishing function $G : \overline{\mathcal{O}} \to K$. To prove this, we consider $\tilde{F}(\mathbf{x}) := F(-\mathbf{x})$ (to eliminate the minus signs; Theorem 2.2 and [1, Theorem 2]) and construct two sequences of continuous functions $\alpha_n(\mathbf{x})$ and $\beta_n(\mathbf{x})$ (see (13) and (14)). Then it is shown in (16) that

$$\tilde{F}\left(\varphi^{(n-1)}\left(\alpha_n(\mathbf{x})\right)\right) = A_n(\mathbf{x}) \cdot B_n(\mathbf{x}) \cdot \frac{\tilde{G}_n(\mathbf{x})}{\tilde{G}_n\left(\varphi(\mathbf{x})\right)}$$

for some (explicit in terms of $\tilde{F}, \alpha_n, \beta_n$ and φ) functions $A_n, B_n, \tilde{G}_n : \overline{\mathcal{O}} \to K$. With some special properties of $\tilde{F}, \alpha_n$ and β_n (see (11), (15), etc.), we prove that the left-hand side converges to $\tilde{F}(\mathbf{x})$ and the right-hand side converges to $\tilde{G}(\mathbf{x})/\tilde{G}(\varphi(\mathbf{x}))$ for some continuous non-vanishing function $\tilde{G} : \overline{\mathcal{O}} \to K$ (Lemma 2.3). Then the desired function $G(\mathbf{x})$ turns out to be $\tilde{G}(-\mathbf{x})$, and the result follows.

2. Proof of the Theorem 1.7

2.1. The "if" Direction. In this section, we give a proof of "$\Leftarrow$" of Theorem 1.7. The primary key of the proof is to reformulate (7) as follows. Recall that $\varphi : \overline{\mathcal{O}} \to \overline{\mathcal{O}}$ is defined by $\varphi(\mathbf{x}) = (\varphi_t(\mathbf{x}_t))_t$ where each $\varphi_t : \overline{\mathcal{O}}_t \to \overline{\mathcal{O}}_t$ is given by

$$\varphi_t(\mathbf{x}_t) := \sum_{i=0}^{\infty} x_{t,i+1}\pi_t^i$$

for any

$$\mathbf{x}_t := \sum_{i=0}^{\infty} x_{t,i}\pi_t^i \in \overline{\mathcal{O}}_t.$$

For $\mathbf{x}, \mathbf{y} \in \overline{\mathcal{O}}$, we denote by

$$\mathbf{x} \equiv \mathbf{y} \pmod{\boldsymbol{\pi}^n}$$

if

$$\mathbf{x}_t \equiv \mathbf{y}_t \pmod{\pi_t^n}$$

for all t. We note that for any $\mathbf{x}, \mathbf{y} \in \overline{\mathcal{O}}$,

$$\mathbf{x} \equiv \mathbf{y} \pmod{\boldsymbol{\pi}^n} \implies \varphi(\mathbf{x}) \equiv \varphi(\mathbf{y}) \pmod{\boldsymbol{\pi}^{n-1}}. \tag{8}$$

Furthermore, if $\mathbf{x} = (\mathbf{x}_t)_t \in (\boldsymbol{\pi}^n - \mathbf{1})^{-1}\mathcal{O}$ satisfying the condition (6), then we have

$$\{\langle \boldsymbol{\pi}^j \mathbf{x} \rangle : 0 \le j \le n-1\} = \{-\varphi^{(j)}(-\mathbf{x}) : 0 \le j \le n-1\}.$$

The identity above follows from expressing each $\mathbf{x}_t = m_t/(\pi_t^n - 1)$ where $0 \le |m_t| < |\pi_t^n - 1|$ (this means that m_t is a positive integer with $0 \le m_t < \pi_t^n - 1$ when $\mathcal{O}_t = \mathbb{Z}$ or a polynomial in A with $\deg m_t < \deg(\pi_t^n - 1) = \deg \pi_t^n$ when $\mathcal{O}_t = A$), and observing that the element $\langle \pi_t \mathbf{x}_t \rangle$ (resp. $-\varphi_t(-\mathbf{x}_t)$) in each component corresponds to shifting the π_t-adic digits of m_t one step right (resp. left). So when j runs through 0 to $n-1$, two sets are in fact identical. To conclude, the formula (7) can be rewritten as follows: For all $n \in \mathbb{N}$ and all $\mathbf{x} \in (\boldsymbol{\pi}^n - \mathbf{1})^{-1}\mathcal{O}$ satisfying the condition (6), we have

$$\prod_{j=0}^{n-1} \boldsymbol{\Gamma}\left(-\varphi^{(j)}(-\mathbf{x})\right) = \prod_{j=0}^{n-1} H\left(-\varphi^{(j)}(-\mathbf{x})\right). \tag{9}$$

To complete the proof of "$\Leftarrow$" of Theorem 1.7, suppose that we are given any continuous non-vanishing function $G : \overline{\mathcal{O}} \to K$. We claim that the following function

$$H(\mathbf{x}) := \mathbf{\Gamma}(\mathbf{x}) \cdot \frac{G(\mathbf{x})}{G\left(-\varphi(-\mathbf{x})\right)}$$

is continuous, non-vanishing, and satisfies the identity (9), whence the proof would be completed. To prove the claim above, we first note that the continuous and non-vanishing properties are clear from the definition of H. To show the formula (9), we further mention that for $\mathbf{x} \in (\boldsymbol{\pi}^n - \mathbf{1})^{-1}\mathcal{O}$ with condition (6), we have

$$\prod_{j=0}^{n-1} \frac{G\left(-\varphi^{(j)}\left(-\mathbf{x}\right)\right)}{G\left(-\varphi^{(j+1)}\left(-\mathbf{x}\right)\right)} = 1 \tag{10}$$

as $-\varphi^{(n)}(-\mathbf{x}) = \mathbf{x}$. Hence, the desired identity is satisfied.

2.2. **The "only if" Direction.** We now proceed to prove "$\Rightarrow$" of Theorem 1.7. For any $n \in \mathbb{N}$ and $\mathbf{x} = (\mathbf{x}_t)_t \in \overline{\mathcal{O}}$, we observe that the following two conditions are equivalent:

- $\mathbf{x} \in (\boldsymbol{\pi}^n - \mathbf{1})^{-1}\mathcal{O}$ and $\mathbf{x}$ satisfies (6).
- $\varphi^{(n)}(-\mathbf{x}) = -\mathbf{x}$ and $|\mathbf{x}_t| < 1$ for each t.

Let $F(\mathbf{x}) := H(\mathbf{x})/\mathbf{\Gamma}(\mathbf{x})$. Then using the reformulation (9), it suffices to prove the following theorem, which is a variant of [1, Theorem 1] in this more general setting. The key idea of the proof traces back to the original arguments of Adolphson.

Theorem 2.1. *Let $F : \overline{\mathcal{O}} \to K$ be a continuous non-vanishing function. Suppose that for all $n \in \mathbb{N}$ and all $\mathbf{x} = (\mathbf{x}_t)_t \in \overline{\mathcal{O}}$ with $|\mathbf{x}_t| < 1$ for each t, we have*

$$\varphi^{(n)}(-\mathbf{x}) = -\mathbf{x} \implies \prod_{j=0}^{n-1} F\left(-\varphi^{(j)}(-\mathbf{x})\right) = 1.$$

Then there exists a continuous non-vanishing function $G : \overline{\mathcal{O}} \to K$ such that

$$F(\mathbf{x}) = \frac{G(\mathbf{x})}{G(-\varphi(-\mathbf{x}))}$$

for all $\mathbf{x} \in \overline{\mathcal{O}}$.

By replacing $F(\mathbf{x}), G(\mathbf{x})$ with $F(-\mathbf{x}), G(-\mathbf{x})$ respectively, Theorem 2.1 is equivalent to the following (cf. [1, Theorem 2]).

Theorem 2.2. *Let $F : \overline{\mathcal{O}} \to K$ be a continuous non-vanishing function. Suppose that for all $n \in \mathbb{N}$ and all $\mathbf{x} = (\mathbf{x}_t)_t \in \overline{\mathcal{O}}$ with $|\mathbf{x}_t| < 1$ for each t, we have*

$$\varphi^{(n)}(\mathbf{x}) = \mathbf{x} \implies \prod_{j=0}^{n-1} F\left(\varphi^{(j)}(\mathbf{x})\right) = 1. \tag{11}$$

Then there exists a continuous non-vanishing function $G : \overline{\mathcal{O}} \to K$ such that

$$F(\mathbf{x}) = \frac{G(\mathbf{x})}{G(\varphi(\mathbf{x}))}$$

for all $\mathbf{x} \in \overline{\mathcal{O}}$.

Proof. Let $\mathbf{b} := (b_t)_t \in \mathcal{O}$ such that for each t,

$$\begin{cases} 0 \le b_t < \pi_t - 1, & \text{if } \mathcal{O}_t = \mathbb{Z}, \\ |b_t| < |\pi_t - 1|, & \text{if } \mathcal{O}_t = A. \end{cases} \tag{12}$$

Recall for any $\mathbf{x} = (\mathbf{x}_t)_t \in \overline{\mathcal{O}}$, we write

$$\mathbf{x}_t := \sum_{i=0}^{\infty} x_{t,i}\pi_t^i \in \overline{\mathcal{O}}_t$$

in π_t-adic expansion. We set

$$\mathbf{x}_{(i)} := (x_{t,i})_t \in \mathcal{O} \quad \text{so that} \quad \mathbf{x} = \sum_{i=0}^{\infty} \mathbf{x}_{(i)}\boldsymbol{\pi}^i.$$

Define $\alpha_n, \beta_n : \overline{\mathcal{O}} \to \overline{\mathcal{O}}$ by

$$\alpha_n(\mathbf{x}) = \frac{1}{\mathbf{1} - \boldsymbol{\pi}^{2n-1}} \left(\mathbf{b} + \mathbf{b}\boldsymbol{\pi} + \cdots + \mathbf{b}\boldsymbol{\pi}^{n-2} + \boldsymbol{\pi}^{n-1} \left(\sum_{i=0}^{n-1} \mathbf{x}_{(i)}\boldsymbol{\pi}^i \right) \right), \tag{13}$$

and

$$\beta_n(\mathbf{x}) = \frac{1}{\mathbf{1} - \boldsymbol{\pi}^{2n-2}} \left(\mathbf{b} + \mathbf{b}\boldsymbol{\pi} + \cdots + \mathbf{b}\boldsymbol{\pi}^{n-2} + \boldsymbol{\pi}^{n-1} \left(\sum_{i=0}^{n-2} \mathbf{x}_{(i)}\boldsymbol{\pi}^i \right) \right). \tag{14}$$

We remark that α_n, β_n are locally constant and hence continuous on $\overline{\mathcal{O}}$. Moreover, note that for any $\mathbf{x} \in \overline{\mathcal{O}}$, we have

$$|\alpha_n(\mathbf{x})_t| < 1 \quad \text{and} \quad |\beta_n(\mathbf{x})_t| < 1$$

for all t by our choice of b_t (12), and

$$\varphi^{(2n-1)}\left(\alpha_n(\mathbf{x})\right) = \alpha_n(\mathbf{x}) \quad \text{and} \quad \varphi^{(2n-2)}\left(\beta_n\left(\varphi(\mathbf{x})\right)\right) = \beta_n\left(\varphi(\mathbf{x})\right). \tag{15}$$

Hence by (11), we obtain

$$\prod_{i=0}^{2n-2} F\left(\varphi^{(i)}\left(\alpha_n(\mathbf{x})\right)\right) = 1 \quad \text{and} \quad \prod_{i=0}^{2n-3} F\left(\varphi^{(i)}\left(\beta_n\left(\varphi(\mathbf{x})\right)\right)\right) = 1.$$

By equating these two expressions, we have

$$F\left(\varphi^{(n-1)}\left(\alpha_n(\mathbf{x})\right)\right) = \frac{\prod_{i=0}^{2n-3} F\left(\varphi^{(i)}\left(\beta_n\left(\varphi(\mathbf{x})\right)\right)\right)}{\prod_{i=0}^{n-2} F\left(\varphi^{(i)}\left(\alpha_n(\mathbf{x})\right)\right) \prod_{i=n}^{2n-2} F\left(\varphi^{(i)}\left(\alpha_n(\mathbf{x})\right)\right)}.$$

By multiplying both the numerator and the denominator of the the right-hand side by $\prod_{i=0}^{n-2} F\left(\varphi^{(i)}\left(\beta_n(\mathbf{x})\right)\right)$, we obtain

$$F\left(\varphi^{(n-1)}\left(\alpha_n(\mathbf{x})\right)\right) = A_n(\mathbf{x}) \cdot B_n(\mathbf{x}) \cdot \frac{G_n(\mathbf{x})}{G_n\left(\varphi(\mathbf{x})\right)} \tag{16}$$

where

$$A_n(\mathbf{x}) := \frac{\prod_{i=0}^{n-2} F\left(\varphi^{(i)}\left(\beta_n(\mathbf{x})\right)\right)}{\prod_{i=0}^{n-2} F\left(\varphi^{(i)}\left(\alpha_n(\mathbf{x})\right)\right)}, \quad B_n(\mathbf{x}) := \frac{\prod_{i=n-1}^{2n-3} F\left(\varphi^{(i)}\left(\beta_n\left(\varphi(\mathbf{x})\right)\right)\right)}{\prod_{i=n}^{2n-2} F\left(\varphi^{(i)}\left(\alpha_n(\mathbf{x})\right)\right)} \tag{17}$$

and

$$G_n(\mathbf{x}) := \left(\prod_{i=0}^{n-2} F\left(\varphi^{(i)}\left(\beta_n(\mathbf{x})\right)\right)\right)^{-1}. \tag{18}$$

The result now follows from the following lemma (cf. [1, Lemmas 1-5]), which will be proved in the next section.

Lemma 2.3. *Let the notations be as above. The following statements hold.*

(1) $\lim_{n\to\infty} F\left(\varphi^{(n-1)}\left(\alpha_n(\mathbf{x})\right)\right) = F(\mathbf{x})$.
(2) $\lim_{n\to\infty} A_n(\mathbf{x}) = 1$.
(3) $\lim_{n\to\infty} B_n(\mathbf{x}) = 1$.
(4) $\lim_{n\to\infty} G_n(\mathbf{x}) = G(\mathbf{x})$, *where G is a continuous non-vanishing function on $\overline{\mathcal{O}}$.*

□

2.3. **Proof of Lemma 2.3.** In this section, we prove Lemma 2.3 by following the approach of [1]. Denote the non-Archimedean absolute value on K by $\|\cdot\|$. We first establish the following lemma.

Lemma 2.4. *For $m \in \mathbb{N}$ and $1 \le i \le m$, let $X_i \in K$ with $\|X_i\| = 1$. Then we have*

$$\left\|\prod_{i=1}^{m} X_i - 1\right\| \le \max_{1\le i\le m} \|X_i - 1\|.$$

Proof. By non-Archimedean property, we have

$$\left\|\prod_{i=1}^{m} X_i - 1\right\| = \left\|\sum_{i=1}^{m}\left(\prod_{j>i} X_j\right)(X_i - 1)\right\| \le \max_{1\le i\le m} \|X_i - 1\|.$$

□

Claim 1. $\lim_{n\to\infty} F\left(\varphi^{(n-1)}\left(\alpha_n(\mathbf{x})\right)\right) = F(\mathbf{x})$ (cf. [1, Lemma 1]).

Proof. Note that

$$\varphi^{(n-1)}\left(\alpha_n(\mathbf{x})\right) \equiv \mathbf{x} \pmod{\boldsymbol{\pi}^n}.$$

Thus, we have that $\varphi^{(n-1)}\left(\alpha_n(\mathbf{x})\right) \to \mathbf{x}$ as $n \to \infty$. By the continuity of F, the result follows. □

Since F is continuous and non-vanishing on the compact set $\overline{\mathcal{O}}$ (which is a finite product of the compact sets $\overline{\mathcal{O}}_t$), there exist $\delta_1, \delta_2 > 0$ such that for all $\mathbf{x} \in \overline{\mathcal{O}}$,

$$\delta_1 \leq \|F(\mathbf{x})\| \leq \delta_2. \tag{19}$$

Moreover, since F is continuous on $\overline{\mathcal{O}}$, it is uniformly continuous. Hence, given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\mathbf{x} \equiv \mathbf{y} \pmod{\boldsymbol{\pi}^{N_\varepsilon}} \implies \|F(\mathbf{x}) - F(\mathbf{y})\| < \delta_1 \varepsilon. \tag{20}$$

Claim 2. $\lim_{n\to\infty} A_n(\mathbf{x}) = 1$ (cf. [1, Lemma 2]).

Proof. Let $\varepsilon > 0$ where we may assume $\varepsilon < 1$. Note that

$$\alpha_n(\mathbf{x}) \equiv \beta_n(\mathbf{x}) \pmod{\boldsymbol{\pi}^{2n-2}}.$$

Thus by (8), for $i = 0, 1, \ldots, n-2$, we have

$$\varphi^{(i)}\left(\alpha_n(\mathbf{x})\right) \equiv \varphi^{(i)}\left(\beta_n(\mathbf{x})\right) \pmod{\boldsymbol{\pi}^n}.$$

So for $n \geq N_\varepsilon$,

$$\varphi^{(i)}\left(\alpha_n(\mathbf{x})\right) \equiv \varphi^{(i)}\left(\beta_n(\mathbf{x})\right) \pmod{\boldsymbol{\pi}^{N_\varepsilon}}.$$

Then (20) implies that

$$\left\| F\left(\varphi^{(i)}\left(\beta_n(\mathbf{x})\right)\right) - F\left(\varphi^{(i)}\left(\alpha_n(\mathbf{x})\right)\right) \right\| < \delta_1 \varepsilon.$$

Set

$$a_{i,n}(\mathbf{x}) := \frac{F\left(\varphi^{(i)}\left(\beta_n(\mathbf{x})\right)\right)}{F\left(\varphi^{(i)}\left(\alpha_n(\mathbf{x})\right)\right)}.$$

By (19) we know $\|F(\mathbf{x})\| \geq \delta_1$ for all $\mathbf{x} \in \overline{\mathcal{O}}$, which shows that

$$\|a_{i,n}(\mathbf{x}) - 1\| < \varepsilon.$$

Since $\varepsilon < 1$, we see from the strong triangle inequality that

$$\|a_{i,n}(\mathbf{x})\| = 1.$$

Therefore, by (17) and Lemma 2.4, we obtain

$$\|A_n(\mathbf{x}) - 1\| = \left\| \prod_{i=0}^{n-2} a_{i,n}(\mathbf{x}) - 1 \right\| \leq \max_{0 \leq i \leq n-2} \|a_{i,n}(\mathbf{x}) - 1\| < \varepsilon.$$

This completes the proof. □

Claim 3. $\lim_{n\to\infty} B_n(\mathbf{x}) = 1$ (cf. [1, Lemma 3]).

Proof. Note that

$$\varphi^{(n)}\left(\alpha_n(\mathbf{x})\right) \equiv \varphi^{(n-1)}\left(\beta_n\left(\varphi(\mathbf{x})\right)\right) \pmod{\boldsymbol{\pi}^{2n-2}}.$$

Thus by (8), for $i = 0, 1, \ldots, n-2$, we have

$$\varphi^{(n+i)}\left(\alpha_n(\mathbf{x})\right) \equiv \varphi^{(n-1+i)}\left(\beta_n\left(\varphi(\mathbf{x})\right)\right) \pmod{\boldsymbol{\pi}^n}.$$

The result now follows from a similar argument as in Claim 2. □

Claim 4. $\lim_{n\to\infty} G_n(\mathbf{x}) = G(\mathbf{x})$ where G is a continuous non-vanishing function on $\overline{\mathcal{O}}$ (cf. [1, Lemmas 4 and 5]).

Proof. We first show that $\{G_n\}_{n=1}^{\infty}$ is uniformly bounded away from zero. That is, there exist $M_1, M_2 > 0$ such that for all $n \in \mathbb{N}$ and all $\mathbf{x} \in \overline{\mathcal{O}}$,

$$M_1 \leq \|G_n(\mathbf{x})\| \leq M_2.$$

Note that by (12),

$$\left|\left(\frac{\mathbf{b}}{\mathbf{1}-\boldsymbol{\pi}}\right)_t\right| = \left|\frac{b_t}{1-\pi_t}\right| < 1 \tag{21}$$

for each t, and

$$\varphi\left(\frac{\mathbf{b}}{\mathbf{1}-\boldsymbol{\pi}}\right) = \frac{\mathbf{b}}{\mathbf{1}-\boldsymbol{\pi}}.$$

Thus by (11),

$$F\left(\frac{\mathbf{b}}{\mathbf{1}-\boldsymbol{\pi}}\right) = 1.$$

By continuity of F, there exists a positive integer N such that for all $\mathbf{y} \in \overline{\mathcal{O}}$ with

$$\mathbf{y} \equiv \frac{\mathbf{b}}{\mathbf{1}-\boldsymbol{\pi}} \pmod{\boldsymbol{\pi}^N},$$

we have

$$\left\|F(\mathbf{y}) - F\left(\frac{\mathbf{b}}{\mathbf{1}-\boldsymbol{\pi}}\right)\right\| = \|F(\mathbf{y}) - 1\| < 1.$$

We note that in this case, $\|F(\mathbf{y})\| = 1$.

Now, notice that

$$\beta_n(\mathbf{x}) \equiv \frac{\mathbf{b}}{\mathbf{1}-\boldsymbol{\pi}} \pmod{\boldsymbol{\pi}^{n-1}}.$$

Hence by (8), for $i = 0, 1, \ldots, n-N-1$,

$$\varphi^{(i)}\left(\beta_n(\mathbf{x})\right) \equiv \varphi^{(i)}\left(\frac{\mathbf{b}}{\mathbf{1}-\boldsymbol{\pi}}\right) = \frac{\mathbf{b}}{\mathbf{1}-\boldsymbol{\pi}} \pmod{\boldsymbol{\pi}^N},$$

which implies that for $i = 0, 1, \ldots, n-N-1$,

$$\left\|F\left(\varphi^{(i)}\left(\beta_n(\mathbf{x})\right)\right)\right\| = 1.$$

Therefore, by (18),

$$\|G_n(\mathbf{x})\| = \left\|\prod_{i=n-N}^{n-2} F\left(\varphi^{(i)}\left(\beta_n(\mathbf{x})\right)\right)\right\|^{-1}.$$

From (19) we know that for all $\mathbf{x} \in \overline{\mathcal{O}}$,

$$\delta_1 \leq \|F(\mathbf{x})\| \leq \delta_2,$$

which implies that

$$\delta_2^{-(N-1)} \leq \|G_n(\mathbf{x})\| \leq \delta_1^{-(N-1)}$$

for all $n \in \mathbb{N}$ and $\mathbf{x} \in \overline{\mathcal{O}}$. This shows that $\{G_n\}_{n=1}^{\infty}$ is uniformly bounded away from zero.

Next, we show that $\{G_n\}_{n=1}^{\infty}$ converges uniformly on $\overline{\mathcal{O}}$. By the completeness of K together with non-Archimedean property, it suffices to show that $\{G_n - G_{n+1}\}_{n=1}^{\infty}$ converges uniformly to 0 on $\overline{\mathcal{O}}$. Since G_{n+1} is non-vanishing, we have

$$G_n - G_{n+1} = G_{n+1}\left(\frac{G_n}{G_{n+1}} - 1\right).$$

And since G_{n+1} is uniformly bounded, it is reduced to showing that given $\varepsilon > 0$, there exists a positive integer N' such that for all $n \geq N'$,

$$\left\|\frac{G_n(\mathbf{x})}{G_{n+1}(\mathbf{x})} - 1\right\| < \varepsilon$$

for all $\mathbf{x} \in \overline{\mathcal{O}}$.

From the definition of $G_n(\mathbf{x})$ (see (18)), we have

$$\frac{G_n(\mathbf{x})}{G_{n+1}(\mathbf{x})} = \frac{\prod_{i=0}^{n-1} F\left(\varphi^{(i)}\left(\beta_{n+1}(\mathbf{x})\right)\right)}{\prod_{i=0}^{n-2} F\left(\varphi^{(i)}\left(\beta_n(\mathbf{x})\right)\right)} = F\left(\beta_{n+1}(\mathbf{x})\right)\frac{\prod_{i=0}^{n-2} F\left(\varphi^{(i+1)}\left(\beta_{n+1}(\mathbf{x})\right)\right)}{\prod_{i=0}^{n-2} F\left(\varphi^{(i)}\left(\beta_n(\mathbf{x})\right)\right)}. \tag{22}$$

Recall we have seen by (11) that

$$F\left(\frac{\mathbf{b}}{\mathbf{1}-\boldsymbol{\pi}}\right) = 1.$$

We have also seen that

$$\beta_{n+1}(\mathbf{x}) \equiv \frac{\mathbf{b}}{\mathbf{1}-\boldsymbol{\pi}} \pmod{\boldsymbol{\pi}^n}.$$

By the continuity of F, there is a positive integer N'' such that for $n \geq N''$, we have

$$\|F\left(\beta_{n+1}(\mathbf{x})\right) - 1\| < \varepsilon.$$

On the other hand, note that

$$\varphi\left(\beta_{n+1}(\mathbf{x})\right) \equiv \beta_n(\mathbf{x}) \pmod{\boldsymbol{\pi}^{2n-2}}.$$

Thus by (8), for $i = 0, 1, \ldots, n-2$, we have

$$\varphi^{(i+1)}\left(\beta_{n+1}(\mathbf{x})\right) \equiv \varphi^{(i)}\left(\beta_n(\mathbf{x})\right) \pmod{\boldsymbol{\pi}^n}.$$

So by (20), for all $n \geq N_\varepsilon$ and all $i = 0, 1, \ldots, n-2$,

$$\left\|F\left(\varphi^{(i+1)}\left(\beta_{n+1}(\mathbf{x})\right)\right) - F\left(\varphi^{(i)}\left(\beta_n(\mathbf{x})\right)\right)\right\| < \delta_1\varepsilon.$$

Set

$$g_{i,n}(\mathbf{x}) := \frac{F\left(\varphi^{(i+1)}\left(\beta_{n+1}(\mathbf{x})\right)\right)}{F\left(\varphi^{(i)}\left(\beta_n(\mathbf{x})\right)\right)}.$$

Since from (19), $\|F(\mathbf{x})\| \geq \delta_1$ for all $\mathbf{x} \in \overline{\mathcal{O}}$, this implies that for all $n \geq N_\varepsilon$ and all $i = 0, 1, \ldots, n-2$,

$$\|g_{i,n}(\mathbf{x}) - 1\| < \varepsilon.$$

We may once again assume $\varepsilon < 1$. Then from the strong triangle inequality,

$$\|g_{i,n}(\mathbf{x})\| = 1.$$

Therefore, by (22) and Lemma 2.4, for $n \geq \max\{N'', N_\varepsilon\}$,

$$\left\| \frac{G_n(\mathbf{x})}{G_{n+1}(\mathbf{x})} - 1 \right\| \leq \max_{0 \leq i \leq n-2} \left\{ \|F(\beta_{n+1}(\mathbf{x})) - 1\|, \|g_{i,n}(\mathbf{x}) - 1\| \right\} < \varepsilon$$

for all $\mathbf{x} \in \overline{\mathcal{O}}$. Hence, the desired property is proved.

Finally, one sees from (18) that each G_n is continuous, so $\{G_n\}_{n=1}^\infty$ converges to a continuous function G. Moreover, since $\{G_n\}_{n=1}^\infty$ is uniformly bounded away from zero, the limit G is non-vanishing. □

3. Generalization

In this section, we consider a variant of Theorem 1.7, which generalizes the domain of the functions but assumes slightly different conditions.

First, assume $\overline{\mathcal{O}}$ is a direct product of finitely many rings $\overline{\mathcal{O}}_t$, where each $\overline{\mathcal{O}}_t$ is the ring of integers of some non-Archimedean local field with a fixed uniformizer. For each t, we let $\pi_t \in \overline{\mathcal{O}}_t$ be a power of the given uniformizer and $S_t \subseteq \overline{\mathcal{O}}_t$ be a complete set of representatives of the quotient ring $\overline{\mathcal{O}}_t/(\pi_t)$. For $\mathbf{x} = (\mathbf{x}_t)_t \in \overline{\mathcal{O}}$, we write

$$\mathbf{x}_t := \sum_{i=0}^{\infty} x_{t,i}\pi_t^i \in \overline{\mathcal{O}}_t,$$

where $x_{t,i} \in S_t$ for each t and i. We similarly define $\varphi : \overline{\mathcal{O}} \to \overline{\mathcal{O}}$ by $\varphi(\mathbf{x}) = (\varphi_t(\mathbf{x}_t))_t$ where each $\varphi_t : \overline{\mathcal{O}}_t \to \overline{\mathcal{O}}_t$ is given by

$$\varphi_t(\mathbf{x}_t) := \sum_{i=0}^{\infty} x_{t,i+1}\pi_t^i.$$

Then Theorem 1.7 can be extended to the following.

Theorem 3.1. *Given any complete non-Archimedean field K, let $\mathbf{\Gamma} : \overline{\mathcal{O}} \to K$ be a continuous non-vanishing function. Then $H : \overline{\mathcal{O}} \to K$ is a function satisfying the following two conditions*

- *H is continuous and non-vanishing;*
- *For all $n \in \mathbb{N}$ and all $\mathbf{x} \in \overline{\mathcal{O}}$ with $\varphi^{(n)}(-\mathbf{x}) = -\mathbf{x}$, we have*

$$\prod_{j=0}^{n-1} \mathbf{\Gamma}\left(-\varphi^{(j)}(-\mathbf{x})\right) = \prod_{j=0}^{n-1} H\left(-\varphi^{(j)}(-\mathbf{x})\right);$$

if and only if

$$H(\mathbf{x}) = \mathbf{\Gamma}(\mathbf{x}) \cdot \frac{G(\mathbf{x})}{G\left(-\varphi(-\mathbf{x})\right)}$$

on $\overline{\mathcal{O}}$, where $G : \overline{\mathcal{O}} \to K$ is any continuous non-vanishing function.

Proof. The "if" part follows immediately from the assumption that $\varphi^{(n)}(-\mathbf{x}) = -\mathbf{x}$ (cf. (10)). For the "only if" part, let $F(\mathbf{x}) := H(\mathbf{x})/\mathbf{\Gamma}(\mathbf{x})$. Then by changing variables (cf. Theorem 2.1 and 2.2), it suffices to prove the following statement (cf. [1, Theorem 3]): Let $F : \overline{\mathcal{O}} \to K$ be a continuous non-vanishing function such that for all $n \in \mathbb{N}$ and all $\mathbf{x} \in \overline{\mathcal{O}}$,

$$\varphi^{(n)}(\mathbf{x}) = \mathbf{x} \implies \prod_{j=0}^{n-1} F\left(\varphi^{(j)}(\mathbf{x})\right) = 1.$$

Then there exists a continuous non-vanishing function $G : \overline{\mathcal{O}} \to K$ such that

$$F(\mathbf{x}) = \frac{G(\mathbf{x})}{G(\varphi(\mathbf{x}))}$$

for all $\mathbf{x} \in \overline{\mathcal{O}}$.

Now we choose any $b_t \in S_t$ for each t and let $\mathbf{b} := (b_t) \in \overline{\mathcal{O}}$. Define $\alpha_n, \beta_n : \overline{\mathcal{O}} \to \overline{\mathcal{O}}$ as (13) and (14). Then the same algebraic manipulation provides the equality (16):

$$F\left(\varphi^{(n-1)}\left(\alpha_n(\mathbf{x})\right)\right) = A_n(\mathbf{x}) \cdot B_n(\mathbf{x}) \cdot \frac{G_n(\mathbf{x})}{G_n\left(\varphi(\mathbf{x})\right)}$$

where A_n, B_n, G_n are defined as (17) and (18). By almost the identical arguments, Lemma 2.3 still holds. The only difference is that, in Claim 4, we do not require the condition (21). Therefore, the result follows in a similar manner. □

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